MATH 246: Chapter 2 Section 6 Part 1 Key Identities
Justin Wyss-Gallifent

1. **Introduction:** Remember where we are: We have a non-homogenous linear differential equation with constant coefficients. We know how to deal with the homogeneous version and we know that all we need to do is get ahold of a single solution to the non-homogeneous version denoted $Y_p(t)$ and then we can construct all solutions to the non-homogeneous version.

   The method of Key Identities will be useful when the forcing function has a certain form, specifically $q(t)e^{\alpha t}\cos(\beta t)$ or $q(t)e^{\alpha t}\sin(\beta t)$ where $q(t)$ is a polynomial with variable t.

   For all of the cases we'll look at will be very simple examples because the computation can get quite difficult. We'll focus on forcing functions like e^{2t} or te^{2t} or $\cos(3t)$ or, rarely, like $t\sin(5t)$.

2. **Notation:** For an expression like $a_2y'' + a_1y' + a_0$ we introduce the notation $L(\square) = a_2\square'' + a_1\square' + a_0\square$. At this point notice that each differential equation has an $L(\square)$ and a characteristic polynomial $p(z)$ which all look alike:

Differential Equation:	$a_2y'' + a_1y' + a_0y = f(t)$
Characteristic Polynomial:	$p(z) = a_2z^2 + a_1z + a_0$
$L(\square)$:	$L(\square) = a_2\square'' + a_1\square' + a_0\square$

   Thus for a differential equation like $y'' - y' - 12y = f(t)$ we have $L(\square) = \square'' - \square' - 12\square$. We can then rewrite the differential equation as $L(y) = f(t)$. The problem of solving the differential equation now means solving $L(y) = f(t)$. Keep this in mind visually, it's important.

   One other thing - note that L is linear in the sense that $\alpha L(X) + \beta L(Y) = L(\alpha X + \beta Y)$.

3. **The Key Identities:** For a complex number z and for $L(\square) = a_2\square'' + a_1\square' + a_0\square$ we have:

   $$\begin{aligned} L(e^{zt}) &= a_2z^2e^{zt} + a_1ze^{zt} + a_0e^{zt} \\ &= (a_2z^2 + a_1z + a_0)e^{zt} \\ &= p(z)e^{zt} \end{aligned}$$

   Next, we can go further, this takes a bit more calculation:

   $$L(te^{zt}) = ... = p(z)te^{zt} + p'(z)e^{zt}$$

   Together these make up the first two Key Identities:

KI#1:	$L(e^{zt}) = p(z)e^{zt}$
KI#2:	$L(te^{zt}) = p(z)te^{zt} + p'(z)e^{zt}$

   We can do more, we can find $L(t^2e^{zt})$ and so on, which form the third, fourth, etc. We won't!

4. **Using The Key Identities:** The idea is this - we rephrase our problem as $L(y) = f(t)$ in which we want to find y. We look at $f(t)$ and find the root (real or complex) which corresponds to it in the Section 4 sense. Call that z. We then write down KI#1 and see if we can fiddle with it until we get $L(\square) = f(t)$. If we can, then $y = \square$ is the answer. If not, then we write down KI#2 and see if we can fiddle with it, or with both of them, until we get $L(\square) = f(t)$. If we can (and in this course we can), then $y = \square$ is the answer. We will heavily make use of the linearity of L.

5. **Examples:**

Example: Consider the differential equation $y'' - y' - 12y = 5e^{2t}$.
Then $L(\Box) = \Box'' - \Box' - 12\Box$ and $p(z) = z^2 - z - 12$. We are now trying to solve $L(y) = 5e^{2t}$.
If we let $z = 2$ then KI#1 becomes:

$$L(e^{zt}) = p(z)e^{zt} \qquad \Longrightarrow \qquad L\left(e^{2t}\right) = -10e^{2t}$$

The goal is $5e^{2t}$ on the right so we then multiply both sides by $-\frac{1}{2}$ and use linearity on the left:

$$L\left(-\frac{1}{2}e^{2t}\right) = 5e^{2t}$$

And hey, $Y_p(t) = -\frac{1}{2}e^{2t}$ is a solution!

Note: Because the fundamental set for the homogeneous version is $\left\{e^{4t}, e^{-3t}\right\}$, the general solution to this nonhomogeneous differential equation is $Y(t) = -\frac{1}{2}e^{2t} + c_1e^{4t} + c_2e^{-3t}$.

Example: Consider the differential equation $y'' - y' - 12y = 3e^{4t}$.
Then $L(\Box) = \Box'' - \Box' - 12\Box$ and $p(z) = z^2 - z - 12$. We are now trying to solve $L(y) = 3e^{4t}$.
If we let $z = 4$ then KI#1 becomes

$$L(e^{zt}) = p(z)e^{zt} \qquad \Longrightarrow \qquad L(e^{4t}) = 0$$

and there's no way to get $3e^{4t}$ on the right. So we look at KI#2, which becomes

$$L(te^{zt}) = p(z)te^{zt} + p'(z)e^{zt} \qquad \Longrightarrow \qquad L(te^{4t}) = 0te^{4t} + 7e^{4t}$$

The goal is $3e^{4t}$ on the right so we multiply both sides by $\frac{3}{7}$ and insert it inside the L via linearity:

$$L\left(\frac{3}{7}te^{4t}\right) = 3e^{4t}$$

And hey, $Y_p(t) = \frac{3}{7}te^{4t}$ is a solution!

Note: Because the fundamental set for the homogeneous version is $\left\{e^{4t}, e^{-3t}\right\}$, the general solution to this nonhomogeneous differential equation is $Y(t) = \frac{3}{7}te^{4t} + c_1e^{4t} + c_2e^{-3t}$.

Example: Consider the differential equation $y''' - y' = 2e^{7t}$. Then $L(\Box) = \Box''' - \Box'$ and $p(z) = z^3 - z$. We are now trying to solve $L(y) = 2e^{7t}$. If we let $z = 7$ then KI#1 becomes

$$L(e^{zt}) = p(z)e^{zt} \qquad \Longrightarrow \qquad L(e^{7t}) = 336e^{7t}$$

The goal is $2e^{7t}$ on the right so we then multiply both sides by $\frac{1}{168}$ and use linearity on the left:

$$L\left(\frac{1}{168}e^{7t}\right) = 2e^{7t}$$

And hey, $Y_p(t) = \frac{1}{168}e^{7t}$ is a solution!

Note: Because the fundamental set for the homogeneous version is $\{1, e^t, e^{-t}\}$, the general solution to this nonhomogeneous differential equation is $Y(t) = \frac{1}{168}e^{7t} + c_1 + c_2e^t + c_3e^{-t}$.

Example: Consider the differential equation $y'' - y' - 12y = 10te^{-2t}$.
Then $L(\square) = \square'' - \square' - 12\square$ and $p(z) = z^2 - z - 12$. We are now trying to solve $L(y) = 10te^{-2t}$.
Let $z = -2$. Notice that $p'(z) = 2z - 1$ and then observe that KI#1 and KE#2 become:

$$L(e^{zt}) = p(z)e^{zt} \qquad \Longrightarrow \qquad L\left(e^{-2t}\right) = -6e^{-2t}$$
$$L(te^{zt}) = p(z)te^{zt} + p'(z)e^{zt} \qquad \Longrightarrow \qquad L\left(te^{-2t}\right) = -6te^{-2t} - 5e^{-2t}$$

The goal is $10te^{-2t}$ on the right so the idea is to cancel the terms with no t on the right:

$$\begin{aligned}
L\left(-\frac{1}{6}e^{-2t}\right) &= e^{-2t} \quad \text{Using KE\#1} \\
L\left(\frac{1}{5}te^{-2t}\right) &= -\frac{6}{5}te^{-2t} - e^{-2t} \quad \text{Using KE\#2} \\
L\left(-\frac{1}{6}e^{-2t} + \frac{1}{5}te^{-2t}\right) &= -\frac{6}{5}te^{-2t} \quad \text{Adding} \\
L\left(\frac{50}{36}e^{-2t} - \frac{5}{3}te^{-2t}\right) &= 10te^{-2t} \quad \text{Multiplying by } -\frac{50}{6}
\end{aligned}$$

and hey, $Y_p(t) = \frac{50}{36}e^{-2t} - \frac{5}{3}te^{-2t}$ is a solution!

Note: Because the fundamental set for the homogeneous version is $\left\{e^{4t}, e^{-3t}\right\}$, the general solution to this nonhomogeneous differential equation is $Y(t) = \frac{50}{36}e^{-2t} - \frac{5}{3}te^{-2t} + c_1e^{4t} + c_2e^{-3t}$.

Example: Consider the differential equation $L(y) = y'' - y' - 12y = 2\sin(3t)$.
Then $L(\Box) = \Box'' - \Box' - 12\Box$ and $p(z) = z^2 - z - 12$. We are now trying to solve $L(y) = 2\sin(3t)$.
Since $\sin(3t) = e^{0t}\sin(3t)$ we let $z = 0 + 3i$ then KI#1 becomes:

$$L(e^{zt}) = p(z)e^{zt} \qquad \Longrightarrow \qquad L\left(e^{(3i)t}\right) = (-21 - 3i)e^{(3i)t}$$

Okay, this is not nearly as nice as the previous problem. First we'll do some rewriting:

$$\begin{aligned}
L\left(e^{(3i)t}\right) &= (-21 - 3i)e^{(3i)t} \\
L\left(\cos(3t) + i\sin(3t)\right) &= (-21 - 3i)(\cos(3t) + i\sin(3t)) \\
L\left(\left(\frac{1}{-21 - 3i}\right)\cos(3t) + i\sin(3t)\right) &= \cos(3t) + i\sin(3t)
\end{aligned}$$

We then clean up the complex fraction inside by multiplying by the reciprocal:

$$\frac{1}{-21 - 3i} = \frac{1}{-21 - 3i}\frac{-21 + 3i}{-21 + 3i} = \frac{-21 + 3i}{441 + 9} = \frac{-21 + 3i}{450} = -\frac{7}{150} + \frac{1}{150}i$$

and rewrite:

$$L\left(\left(-\frac{7}{150} + \frac{1}{150}i\right)(\cos(3t) + i\sin(3t))\right) = \cos(3t) + i\sin(3t)$$

Okay now understand something - if $L(A + Bi) = C + Di$ then $L(A) = C$ and $L(B) = D$, all by linearity. Since the right side we want only has the sine (the imaginary part), we only need the imaginary part inside on the left. Slightly less vaguely if we FOIL and simplify the inside left part, only paying attention to the imaginary part:

$$L\left(\text{REAL PART} + \left(-\frac{7}{150}\sin(3t) + \frac{1}{150}\cos(3t)\right)i\right) = \cos(3t) + i\sin(3t)$$

On the right, the $\sin(3t)$ part is the part we want so let's match up the imaginary parts:

$$L\left(-\frac{7}{150}\sin(3t) + \frac{1}{150}\cos(3t)\right) = \sin(3t)$$

Then we multiply by 2 to get:

$$L\left(-\frac{7}{75}\sin(3t) + \frac{1}{75}\cos(3t)\right) = 2\sin(3t)$$

And hey, $Y_p(t) = -\frac{7}{75}\sin(3t) + \frac{1}{75}\cos(3t)$ is a solution!
Note: Because the fundamental set for the homogeneous version is $\left\{e^{4t}, e^{-3t}\right\}$, the general solution to this nonhomogeneous differential equation is $Y_p(t) = -\frac{7}{75}\sin(3t) + \frac{1}{75}\cos(3t) + c_1e^{4t} + c_2e^{-3t}$.

6. **Summary:** The problem can get worse. If there is a t in front and the z is a root then we need the third Key Identity. If the root has higher multiplicity or there is a higher degree polynomial like a t^2 in front then we need even more Key Identities. It's fun, really.