MATH 246: Chapter 2 Section 8 Motion
Justin Wyss-Gallifent

1. **Introduction**

   Important: Positive is up and negative is down.

   Imagine a spring hanging with no weight on it. We then attach a mass m which stretches the spring a distance of $y_R < 0$. We are now at the rest point. At this point the force of gravity is mg (negative since $g < 0$, think $g = -9.8$ if it helps) and the force of the spring by Hooke's Law is $-ky_R$ (the spring force is upwards so we negate against $y_R < 0$). Consequently $mg + (-ky_R) = 0$ and so $mg = ky_R$.

   Now then, imagine the object and spring system is in motion and at any time t the displacement from the rest point is given by $y(t)$. At any instant now there could be multiple forces acting on the object:

Gravity	mg	Acting downwards with $g < 0$.
Spring	$-k(y + y_R)$	Acting against the displacement.
Damping	$-\gamma y'$	Acting against and proportional to velocity, here $\gamma > 0$.
External	$f(t)$	Some other external force.

   When we put these all together we get:

   $$\begin{aligned} F_{Tot} &= F_{Grav} + F_{Spring} + F_{Damping} + F_{External} \\ my'' &= mg - k(y + y_R) - \gamma y' + f(t) \\ my'' &= ky_r - ky - ky_R - \gamma y' + f(t) \\ my'' &= -ky - \gamma y' + f(t) \end{aligned}$$

   We finall rewrite this as:

   $$my'' + \gamma y' + ky = f(t)$$

   If this doesn't look familiar then you've been asleep for several weeks!

2. **A Few Notes**

   (a) In the Metric system we may either have length, time, mass and force in meters, seconds, kilograms and newtons ($newton = kg \cdot meter/s^2$) respectively, or in centimeters, seconds, grams and dynes ($dyne = g \cdot cm/s^2$) respectively. In the British system we may have feet, seconds, slugs and pounds ($lb = slug \cdot ft/s^2$). Note also in the British system that weight is also in pounds with $lb = slug \cdot gravity$.

   (b) If k is not given we may need to find it using $mg = ky_R$. We would be given the mass m of the object and the displacement y_R. We can then find k. For example if an object of mass 2 kilograms displace a spring 0.5 meters downwards then $(2)(-9.8) = k(-0.5)$.

   (c) If γ is not given we may need to find it using $F_{Damping} = \gamma y'$. We would be given the damping force for a certain velocity. For example if a mass traveling at $0.1m/s$ upwards invokes a damping force of $0.3N$ downwards then $-0.3 = -\gamma(0.1)$.

3. **Unforced and Undamped**

The simplest situation is when there is no external force and no damping. In this case we have $my'' + ky = 0$. The characteristic polynomial has roots $0 \pm i\sqrt{\frac{k}{m}}$ and so the solution is given by

$$y(t) = c_1 \cos\left(t\sqrt{\frac{k}{m}}\right) + c_2 \sin\left(t\sqrt{\frac{k}{m}}\right)$$

This can be rewritten using the Subtraction Formula for Cosine as

$$y(t) = A\cos\left(t\sqrt{\frac{k}{m}} - \delta\right)$$

where $A = \sqrt{c_1 + c_2}$ is the amplitude and δ satisfies $\cos\delta = \frac{c_1}{A}$ and $\sin\delta = \frac{c_2}{A}$. The graph of this makes good sense for a spring that's bouncing up and down forever.

Example: A mass of $0.4kg$ hangs from a spring with coefficient $k = 0.1$. It is pulled down $0.2m$ from resting and released at a rate of $0.3m/s$ downwards.

We have $0.4y'' + 0.1y = 0$ or $y'' + 0.25y = 0$ with $y(0) = -0.2$ and $y'(0) = -0.3$.

The characteristic polynomial has roots $0 \pm i\sqrt{\frac{0.1}{0.4}} = 0 \pm 0.5i$. The general solution is then

$$y(t) = c_1 \cos 0.5t + c_2 \sin 0.5t$$

For the initial value observe $y'(t) = -0.5c_1 \sin 0.5t + 0.5c_2 \cos 0.5t$ and so $y(0) = c_1 = -0.2$ and $y'(0) = 0.5c_2 = -0.3$ so $c_2 = -0.6$. This gives us the specific solution

$$y(t) = -0.2\cos 0.5t - 0.6\sin 0.5t$$

The amplitude is $A = \sqrt{(-0.2)^2 + (0.6)^2} = \sqrt{0.4} \approx 0.63$ We can even draw a nice sketch.

Sketch omitted, but this starts at $(0. - 0.2)$ with a slope of -0.3 and settles into a regular oscillation.

4. **Unforced with Damping**

Now we have $my'' + \gamma y' + ky = 0$. The characteristic polynomial has roots $-\frac{\gamma}{2m} \pm \frac{\sqrt{\gamma^2-4mk}}{2m}$. the behavior of this depends strongly on $\gamma^2 - 4mk$.

(a) **Underdamped:** When $\gamma^2 - 4mk < 0$ (meaning the damping coefficient is small) we have complex roots and our solution has both exponential and trigonometric components. The function starts out oscillating but then the amplitude drops, limiting to zero.

Example: A mass of $0.4kg$ hangs from a spring with coefficient $k = 0.1$ in a fluid with damping coefficient $\gamma = 0.15$. It is pulled down $0.2m$ from resting and released at a rate of $0.3m/s$ downwards.

We have $0.4y'' + 0.15y' + 0.1y = 0$ or $y'' + 0.375y' + 0.25y = 0$ with $y(0) = -0.2$ and $y'(0) = -0.3$.

The characteristic polynomial has roots $z = \frac{-0.15 \pm \sqrt{(0.15)^2 - 4(0.4)(0.1)}}{2(0.4)} = -0.1875 \pm \frac{\sqrt{0.1375}}{0.8} i$ The general solution is then

$$y(t) = c_1 e^{-0.1875t} \cos\left(\frac{\sqrt{0.1375}}{0.8} t\right) = c_2 e^{-0.1875t} \sin\left(\frac{\sqrt{0.1375}}{0.8} t\right)$$

The initial value calculation is much more complicated here but we can draw a reasonable sketch anyway to make sure we understand what a function like this looks like.

Sketch omitted, but this starts at $(0. -0.2)$ with a slope of -0.3 and settles into an oscillation wich reduces over time and limits to zero.

(b) **Critically Damped:** When $\gamma^2 - 4mk = 0$ we have a real root of multiplicity two. This is the special critically damped case. It corresponds to the smallest possible γ for which the oscillation stops.

Example: A mass of $0.4kg$ hangs from a spring with coefficient $k = 0.15625$ in a fluid with damping coefficient $\gamma = 0.5$. It is pulled down $0.7m$ and released with zero velocity.

We have $0.4y'' + 0.5y' + 0.15625y = 0$ or $y'' + 1.25y' + 0.390625y = 0$ with $y(0) = -0.7$ and $y'(0) = 0$.

The characteristic polynomial $z^2 + 1.25y + 0.390625$ has a single root of multiplicity 2 as it factors as $(z + 0.625)^2$. The general solution is then

$$y(t) = c_1 e^{-0.625t} + c_2 t e^{-0.625t}$$

The initial value calculation is much more complicated here but we can draw a reasonable sketch anyway to make sure we understand what a function like this looks like.

Sketch omitted, but this starts at $(0. - 0.7)$ with a slope of 0 and heads directly but asymptotically to the t-axis.

(c) **Overdamped:** When $\gamma^2 - 4mk > 0$ we have two real roots and the system is overdamped.

Example: A mass of $0.4kg$ hangs from a spring with coefficient $k = 0.1$ in a fluid with damping coefficient $\gamma = 0.5$. It is pushed up $0.6m$ and released with zero velocity.

We have $0.4y'' + 0.5y' + 0.1y = 0$ or $y'' + 1.25y' + 0.25y = 0$ with $y(0) = 0.6$ and $y'(0) = 0$.

The characteristic polynomial $z^2 + 1.25y + 0.25$ factors as $(z + 1)(z + 0.25)$ with roots $-1, -0.25$ and hence the general solution is then

$$y(t) = c_1 e^{-t} + c_2 e^{-0.25}$$

For the initial value observe $y'(t) = -c_1 e^{-t} - 0.25c_2 e^{-0.25t}$ and so $y(0) = c_1 + c_2 = 0.6$ and $y'(0) = -c_1 - 0.25c_2$. This yields $c_2 = 0.8$ and $c_1 = -0.2$. This gives us the specific solution

$$y(t) = -0.2e^{-t} + 0.8e^{-0.25t}$$

We can even draw a nice sketch.

Sketch omitted, but this starts at (0.0.6) with a slope of 0 and heads directly but asymptotically to the t-axis.

(d) **A Note on Critically Damped vs. Overdamped:** These two functions look very similar. The critical thing to note is that a damped system oscillates, an overdamped system doesn't, and a critically damped system doesn't either but lies right on the edge of the other two.

5. **Forced**

With forced motions $f(t) \neq 0$ and all bets are off. We know we need to find a particular solution Y_p and then add the general solution to the homogeneous system. This makes sense because the system is governed by both that forcing function and the usual spring motion stuff. In general the behavior will look springy at the start, although the damping might suppress this a bit, and in the long term (assuming damping) will look as if only the forcing function is acting on it.

Example: A mass of $0.4kg$ hangs from a spring with coefficient $k = 0.1$ in a fluid with damping coefficient $\gamma = 0.15$. It is pulled up $0.2m$ from resting and released at a rate of $0.3m/s$ downwards. An additional external force $f(t) = 0.3$ acts downwards on it.

We have $0.4y'' + 0.15y' + 0.1y = -0.3$ or $y'' + 0.375y' + 0.25y = -0.75$ with $y(0) = -0.2$ and $y'(0) = -0.3$.

The Method of Undetermined Coefficients gives us one solution $Y_p(t) = 3$. The homogeneous version is an earlier problem with general solution

$$c_1 e^{-0.1875t} \cos\left(\frac{\sqrt{0.1375}}{2} t\right) = c_2 e^{-0.1875t} \sin\left(\frac{\sqrt{0.1375}}{2} t\right)$$

and hence the general solution to our forced problem is

$$Y(t) = -3 + c_1 e^{-0.1875t} \cos\left(\frac{\sqrt{0.1375}}{2} t\right) = c_2 e^{-0.1875t} \sin\left(\frac{\sqrt{0.1375}}{2} t\right)$$

Notice that in the long term the exponentials take complicated part to zero so that $\lim_{t\to\infty} Y(t) = 3$. So in the long term only the forcing function remains acting on it.

The initial value calculation is much more complicated here but we can draw a reasonable sketch anyway to make sure we understand what a function like this looks like.

Sketch omitted, but this starts at $(0, 0.2)$ with slope -0.3 and oscillates with reducing amplitude as it settles down and limits to $y = -3$.