

MATH 246: Chapter 2 section 9 Laplace Transforms
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1. **Introduction:** Laplace transforms are a way of changing one function into another function. Basically we start with a function of t and change it to a function of s . We can also do the reverse. The Laplace transform has some really useful properties which will help us solve initial value problems.
2. **Formal Definition:** If $f(t)$ is a function then the Laplace transform of this function is formally defined by:

$$\mathcal{L}[y(t)](s) = \int_0^{\infty} y(t)e^{-st} dt = \lim_{b \rightarrow \infty} \int_0^b y(t)e^{-st} dt$$

I'll write $\mathcal{L}[y(t)]$ without the (s) because the (s) looks like multiplication, which it isn't. This formal definition is used to build a set of rules and the rules are what we'll use. Here's an example with the definition though:

Example: If $y(t) = 1$ then we get:

$$\begin{aligned}\mathcal{L}[1] &= \int_0^{\infty} 1e^{-st} dt \\ &= \lim_{b \rightarrow \infty} \int_0^b e^{-st} dt \\ &= \lim_{b \rightarrow \infty} \left. -\frac{1}{s}e^{-st} \right|_0^b \\ &= \lim_{b \rightarrow \infty} \left(-\frac{1}{s}e^{-sb} + \frac{1}{s}e^{-s(0)} \right) \\ &= \frac{1}{s}\end{aligned}$$

Thus $\mathcal{L}[1] = \frac{1}{s}$.

3. Function Rules

Using this same approach we can prove the following rules for common functions:

Zero:	$\mathcal{L}[0] = 0$
For constants:	$\mathcal{L}[c] = \frac{c}{s}$
For powers $n = 1, 2, \dots$:	$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$
For exponents:	$\mathcal{L}[e^{at}] = \frac{1}{s-a}$
For cosine:	$\mathcal{L}[\cos(bt)] = \frac{s}{s^2+b^2}$
For sine:	$\mathcal{L}[\sin(bt)] = \frac{b}{s^2+b^2}$
For powers:	$\mathcal{L}[e^{at}t^n] = \frac{n!}{(s-a)^{n+1}}$
For cosine:	$\mathcal{L}[e^{at}\cos(bt)] = \frac{s-a}{(s-a)^2+b^2}$
For sine:	$\mathcal{L}[e^{at}\sin(bt)] = \frac{b}{(s-a)^2+b^2}$
Linearity:	$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$

When a and b are constants and f and g are functions.

Notation: Sometimes if a function is denoted $y(t)$ then we use $Y(s)$ instead of $\mathcal{L}[y(t)]$ for convenience. For example from the table if $y(t) = e^{2t}$ then $Y(s) = \frac{1}{s-2}$.

4. Examples Here are some brief examples:

Example: $\mathcal{L}[t^5] = \frac{5!}{s^6} = \frac{120}{s^6}$.
Thus if $y(t) = t^5$ then $Y(s) = \frac{120}{s^6}$.

Example: $\mathcal{L}[3\sin(5t) + 7t^2] = 3\mathcal{L}[\sin(5t)] + 7\mathcal{L}[t^2] = 3\left(\frac{5}{s^2+25}\right) + 7\left(\frac{2}{s^3}\right)$
Thus if $y(t) = 3\sin(5t) + 7t^2$ then $Y(s) = 3\left(\frac{5}{s^2+25}\right) + 7\left(\frac{2}{s^3}\right)$.

Example: $\mathcal{L}[e^{2t}t^7] = \frac{7!}{(s-2)^8}$
Thus if $y(t) = e^{2t}t^7$ then $Y(s) = \frac{7!}{(s-2)^8}$.

5. **Reversing** If we start with $Y(s)$ we can also work in reverse. Here are some examples with comments because sometimes we need to manipulate the function first:

Example: If $Y(s) = \frac{2}{s}$ then $y(t) = 2$.

Example: If $Y(s) = \frac{1}{s^5}$ then first we rewrite $Y(s) = \frac{1}{4!} \left(\frac{4!}{s^5}\right)$ and then we see $y(t) = \frac{1}{24}t^4$.

Example: If $Y(s) = \frac{1}{s+4}$ we think of it as $Y(s) = \frac{1}{s-(-4)}$ then $y(t) = e^{-4t}$.

Example: If $Y(s) = \frac{-18}{s^2+9}$ then first rewrite to get $Y(s) = -6 \left(\frac{3}{s^2+9}\right)$ and then we see $y(t) = -6 \sin(3t)$.

Example: If $Y(s) = \frac{2}{s^2+s}$ then first we see $Y(s) = \frac{2}{s(s+1)}$ and then we need to rewrite with partial fractions first and then following this we need a bit more rewriting to fit the formulas so $Y(s) = \frac{2}{s} + \frac{2}{s+1} = \frac{2}{s} + 2 \left(\frac{1}{s-(-1)}\right)$, and then we see $y(t) = 2 + 2e^{-t}$.

Example: If $Y(s) = \frac{4s+3}{s^2+25}$ then we need to break it up and rewrite a little to fit the formulas:

$Y(s) = \frac{4s+3}{s^2+25} = \frac{4s}{s^2+25} + \frac{3}{s^2+25} = 4 \left(\frac{s}{s^2+25}\right) + \frac{3}{5} \left(\frac{5}{s^2+25}\right)$ and then we see that

$y(t) = 4 \cos(5t) + \frac{3}{5} \sin(5t)$.

Example: If $Y(s) = \frac{s+1}{s^2-4s+5}$ then the denominator doesn't factor so instead we complete the square and then do a bit more rewriting to get $Y(s) = \frac{s+1}{(s-2)^2+1} = \frac{s-2}{(s-2)^2+1} + \frac{3}{(s-2)^2+1}$ and then we see that $y(t) = e^{2t} \cos(t) + 3e^{2t} \sin(t)$.

6. Derivative Rules

It turns out that the Laplace transfer is nice with derivatives of functions too, giving us the following pattern for an unknown $y(t)$:

$$\begin{aligned}\mathcal{L}[y'(t)] &= s\mathcal{L}[y(t)] - y(0) &&= sY(s) - y(0) \\ \mathcal{L}[y''(t)] &= s^2\mathcal{L}[y(t)] - sy(0) - y'(0) &&= s^2Y(s) - sy(0) - y'(0) \\ \mathcal{L}[y'''(t)] &= s^3\mathcal{L}[y(t)] - s^2y(0) - sy'(0) - y''(0) &&= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\ &\dots\text{and so on}\dots\end{aligned}$$

For example if $y(t)$ is unknown but we know $y(0) = 7$ and $y'(0) = -3$ then the second rule tells us that $\mathcal{L}[y''] = s^2Y(s) - s(7) - (-3)$. We'll see very soon why this is significant.

7. Solving Initial Value Problems

When dealing with an initial value problem our approach will be the following:

- Take the Laplace transform of each side.
- Apply the rules for functions and for derivatives to eliminate all the t , all the derivatives and substitute all the initial values.
- Solve the result for $Y(s)$.
- Reverse the Laplace transform to get the solution $y(t)$.

Example: Suppose we have $y' = 3$ with $y(0) = 1$. We do the following:

$$\begin{aligned}y' &= 3 \\ \mathcal{L}[y'] &= \mathcal{L}[3] \\ sY(s) - y(0) &= \frac{3}{s} \\ sY(s) - 1 &= \frac{3}{s} \\ sY(s) &= \frac{3}{s} + 1 \\ Y(s) &= \frac{3}{s^2} + \frac{1}{s} \\ y(t) &= 3t + 1\end{aligned}$$

And we've solved it! Notice that you really need to understand how the various tables are being used here. The Laplace transform table is used at the beginning and end and the derivative rules are also used early on.

Example: Suppose we have $y'' - 2y' - 3y = 0$ with $y(0) = 1$ and $y'(0) = 4$. We do the following:

$$\begin{aligned}
 y'' - 2y' - 3y &= 0 \\
 \mathcal{L}[y''] - 2\mathcal{L}[y'] - 3\mathcal{L}[y] &= \mathcal{L}[0] \\
 (s^2Y(s) - sy(0) - y'(0)) - 2(sY(s) - y(0)) - 3Y(s) &= 0 \\
 s^2Y(s) - s - 4 - 2sY(s) + 2 - 3Y(s) &= 0 \\
 Y(s)(s^2 - 2s - 3) - s - 2 &= 0 \\
 Y(s)(s^2 - 2s - 3) &= s + 2 \\
 Y(s) &= \frac{s + 2}{s^2 - 2s - 3} \\
 Y(s) &= \frac{s + 2}{(s - 3)(s + 1)}
 \end{aligned}$$

Now we need to do some manipulation with partial fractions:

$$\begin{aligned}
 \frac{s + 2}{(s - 3)(s + 1)} &= \frac{A}{s - 3} + \frac{B}{s + 1} \\
 s + 2 &= A(s + 1) + B(s - 3)
 \end{aligned}$$

At this point $s = -1$ gives us $B = -1/4$ and $s = 3$ gives us $A = 5/4$. Back to our problem with the most recent line rewritten:

$$\begin{aligned}
 Y(s) &= \frac{s + 2}{(s - 3)(s + 1)} \\
 Y(s) &= \frac{5/4}{s - 3} + \frac{-1/4}{s + 1} \\
 Y(s) &= \frac{5}{4} \left(\frac{1}{s - 3} \right) - \frac{1}{4} \left(\frac{1}{s - (-1)} \right) \\
 y(t) &= \frac{5}{4}e^{3t} - \frac{1}{4}e^{-t}
 \end{aligned}$$

Example: Suppose we have $y'' + 4y = 2t$ with $y(0) = 1$ and $y'(0) = 0$. We do the following:

$$\begin{aligned}
 y'' + 4y &= 2t \\
 \mathcal{L}[y''] + 4\mathcal{L}[y] &= \mathcal{L}[2t] \\
 s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= \frac{2}{s^2} \\
 s^2Y(s) - s - 0 + 4Y(s) &= \frac{2}{s^2} \\
 Y(s)(s^2 + 4) &= \frac{2}{s^2} + s \\
 Y(s) &= \frac{2}{s^2(s^2 + 4)} + \frac{s}{s^2 + 4}
 \end{aligned}$$

This doesn't look so nice. The second part is okay (it's from \cos) but the first part is not in our table. Instead we need to break it up with partial fractions:

$$\begin{aligned}
 \frac{2}{s^2(s^2 + 4)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 4} \\
 2 &= As(s^2 + 4) + B(s^2 + 4) + (Cs + D)s^2 \\
 2 &= (A + C)s^3 + (B + D)s^2 + 4As + 4B
 \end{aligned}$$

Comparing coefficients gives us $A + C = 0$, $B + D = 0$, $4A = 0$ and $4B = 2$ so that $B = 1/2$, $A = 0$, $D = -1/2$ and $C = 0$ and so back our process with the most recent line rewritten:

$$\begin{aligned}
 Y(s) &= \frac{2}{s^2(s^2 + 4)} + \frac{s}{s^2 + 4} \\
 Y(s) &= \frac{1/2}{s^2} + \frac{-1/2}{s^2 + 4} + \frac{s}{s^2 + 4} \\
 Y(s) &= \frac{1}{2} \left(\frac{1}{s^2} \right) - \frac{1}{4} \left(\frac{2}{s^2 + 4} \right) + \frac{s}{s^2 + 4} \\
 y(t) &= \frac{1}{2}t - \frac{1}{4} \sin(2t) + \cos(2t)
 \end{aligned}$$

Compare this to before where we'd need to find the general solution to the homogeneous version of the differential equation, also find a specific solution to the nonhomogeneous version, add them, then use the initial values to find the constants. This way is significantly faster.

Sometimes it's good practice just to do the first part of a problem:

Example: Find the Laplace Transform of the solution to the initial value problem $y'' + y' - 3y = t + e^{2t} \cos(3t)$ with $y(0) = -1$ and $y'(0) = 2$.

Here all we need to get to is $Y(s)$. We do the following:

$$\begin{aligned}y'' + y' - 3y &= t + e^{2t} \cos(3t) \\ \mathcal{L}[y''] + \mathcal{L}[y'] - 3\mathcal{L}[y] &= \mathcal{L}[t] + \mathcal{L}[e^{2t} \cos(3t)] \\ s^2 Y(s) - sy(0) - y'(0) + sY(s) - y(0) - 3Y(s) &= \frac{s-2}{(s-2)^2 + 9} \\ s^2 Y(s) + s - 2 + sY(s) + 1 - 3Y(s) &= \frac{2-s}{(s-2)^2 + 9} \\ Y(s)(s^2 + s - 3) + s - 1 &= \frac{2-s}{(s-2)^2 + 9} \\ Y(s)(s^2 + s - 3) &= \frac{2-s}{(s-2)^2 + 9} + 1 - s \\ Y(s) &= \frac{2-s}{((s-2)^2 + 9)(s^2 + s - 3)} + \frac{1-s}{s^2 + s - 3}\end{aligned}$$

To finish, this would need to undergo a partial fractions decomposition and then the rules would need to be applied.

8. **Step Functions** The most basic step function is the function which returns 0 up until (but not including) $t = 0$ and then 1 after that. More specifically we have

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

There are other options. If we want to use a value other than 0 we denote it $u_c(t)$:

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

Step functions are useful because they turn other functions on and off. For example the product function $u_\pi(t) \sin(t - \pi)$ is 0 for $t < \pi$ and $\sin(t - \pi)$ for $t \geq \pi$.

It may seem odd that we have $\sin(t - \pi)$ here rather than just $\sin(t)$ but there's a reason why this will usually happen. When a function "kicks in" at a certain t -value this usually means that at that t -value the function begins as though 0 were plugged into it. So for example $u_\pi(t) \sin(t - \pi)$ equals 0 until $t = \pi$ at which point the $\sin(t - \pi)$ part starts behaving as if 0 were plugged in (because of the $t - \pi$ in there).

Example: Suppose a function does nothing until $t = \pi/4$ and then starts behaving like the sine function, meaning like the sine function does at $t = 0$. This new function would be $u_{\pi/4}(t) \sin(t - \pi/4)$.

Example: Suppose a function does nothing until $t = 3$ and then starts behaving like the exponential function e^t , meaning like the exponential function does at $t = 0$. This new function would be $u_3(t)e^{t-3}$.

9. Laplace Transforms and Step Functions

They have the following Laplace transform related behavior:

$$\mathcal{L}[u_c(t)f(t - c)] = e^{-cs} \mathcal{L}[f(t)]$$

Note: For the forward direction think of pulling out the $u_c(t)$, it becomes e^{-cs} and then all $t - c$ inside are replaced by just t , then continue:

Example: $\mathcal{L}[u_3(t)(t - 3)^5] = e^{-3s} \mathcal{L}[t^5] = e^{-3s} \left(\frac{5!}{s^6}\right)$

Example: $\mathcal{L}[u_\pi(t) \sin(t - \pi)] = e^{-\pi s} \mathcal{L}[\sin(t)] = e^{-\pi s} \left(\frac{1}{s^2 + 1}\right)$

Example: $\mathcal{L}[u_2(t)e^{4(t-2)}(t - 2)^5] = e^{-2s} \mathcal{L}[e^{4t}t^5] = e^{-2s} \left(\frac{5!}{(s-4)^6}\right)$

Note: For the backwards direction think if $Y(s) = e^{-cs}J(s)$ then first find $j(t)$ corresponding to $J(s)$, replace the t by $t - c$ and put a $u_c(t)$ in front.

Example: If $Y(s) = e^{-5s} \left(\frac{s}{s^2 + 49}\right)$ then we first find $\cos(7t)$ corresponding to $\frac{s}{s^2 + 49}$, we replace the t by $t - 5$ and put $u_5(t)$ in front, yielding $y(t) = u_5(t) \cos(7(t - 5))$.

Example: If $Y(s) = e^{3s} \left(\frac{5!}{s^6}\right)$ then we first find t^5 corresponding to $\frac{5!}{s^6}$, we replace the t by $t - (-3)$ and put $u_{(-3)}(t)$ in front, yielding $y(t) = u_{(-3)}(t)(t + 3)^5$.

Example: If $Y(s) = e^{-5s} \left(\frac{6!}{(s-3)^7}\right)$ then we first find $e^{3t}t^6$ corresponding to $\frac{6!}{(s-3)^7}$, we replace the t by $t - 5$ and put $u_5(t)$ in front, yielding $y(t) = u_5(t)e^{3(t-5)}(t - 5)^7$.

This can then be tied into initial value problems.

Example: Suppose $y' - 2y = f(t)$ where

$$f(t) = \begin{cases} 0 & t < 3 \\ t - 3 & t \geq 3 \end{cases}$$

and where $y(0) = 0$.

We first note that $f(t) = u_3(t)(t - 3)$ and then proceed:

$$y' - 2y = u_3(t)(t - 3)$$

$$\mathcal{L}[y'] - 2\mathcal{L}[y] = \mathcal{L}[u_3(t)(t - 3)]$$

$$s\mathcal{L}[y] - y(0) - 2\mathcal{L}[y] = e^{-3s}\mathcal{L}[t]$$

$$sY(s) - 2Y(s) = e^{-3s}\left(\frac{1}{s^2}\right)$$

$$Y(s)(s - 2) = e^{-3s}\left(\frac{1}{s^2}\right)$$

$$Y(s) = e^{-3s}\left(\frac{1}{s^2(s - 2)}\right)$$

...Partial Fractions Not Shown...

$$Y(s) = e^{-3s}\left(\frac{-1/4}{s} - \frac{1/2}{s^2} + \frac{1/4}{s - 2}\right)$$

Yields: $-\frac{1}{4} - \frac{1}{2}t + \frac{1}{4}e^{2t}$

$$y(t) = u_3(t)\left(-\frac{1}{4} - \frac{1}{2}(t - 3) + \frac{1}{4}e^{2(t-3)}\right)$$

Example: Suppose $y'' - y' - 2y = f(t)$ where

$$f(t) = \begin{cases} 0 & t < 3 \\ 7 & t \geq 3 \end{cases}$$

and where $y(0) = 0$ and $y'(0) = -2$.

We first note that $f(t) = 7u_3(t)$ and then proceed:

$$\begin{aligned} y'' - y' - 2y &= 7u_3(t) \\ \mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] &= 7\mathcal{L}[u_3(t)] \\ s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 2Y(s) &= 7e^{-3t} \\ s^2Y(s) - s + 2 - sY(s) - 2Y(s) &= 7e^{-3t} \\ Y(s)(s^2 - s - 2) - s + 2 &= 7e^{-3t} \\ Y(s)(s^2 - s - 2) &= 7e^{-3t} + s - 2 \\ Y(s) &= 7e^{-3t} \left(\frac{1}{s^2 - s - 2} \right) + \frac{s - 2}{s^2 - s - 2} \\ Y(s) &= 7e^{-3t} \left(\frac{1}{(s - 2)(s + 1)} \right) + \frac{1}{s + 1} \\ &\dots\text{Partial Fractions Not Shown}\dots \\ Y(s) &= 7e^{-3t} \left(\frac{1/3}{s - 2} - \frac{1/3}{s + 1} \right) + \frac{1}{s + 1} \\ &\qquad\qquad\qquad \text{Yields: } \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} \\ y(t) &= 7u_3(t) \left(\frac{1}{3}e^{2(t-3)} - \frac{1}{3}e^{-1(t-3)} \right) + e^{-t} \end{aligned}$$

Example: Suppose $y'' - y' = f(t)$ where

$$f(t) = \begin{cases} 0 & t < \pi/4 \\ \cos(t - \pi/4) & t \geq \pi/4 \end{cases}$$

and where $y(0) = 0$ and $y'(0) = 0$. We first note that $f(t) = u_{\pi/4}(t) \cos(t - \pi/4)$ and then proceed:

$$y'' - y' = u_{\pi/4} \cos(t - \pi/4)$$

$$\mathcal{L}[y''] - \mathcal{L}[y'] = \mathcal{L}[u_{\pi/4} \cos(t - \pi/4)]$$

$$(s^2 Y(s) - sy(0) - y'(0)) - (sY(s) - y(0)) = e^{-(\pi/4)s} \mathcal{L}[\cos(t)]$$

$$s^2 Y(s) - sY(s) = e^{-(\pi/4)s} \left(\frac{s}{s^2 + 1} \right)$$

$$Y(s)(s^2 - s) = e^{-(\pi/4)s} \left(\frac{s}{s^2 + 1} \right)$$

$$Y(s) = e^{-(\pi/4)s} \left(\frac{s}{(s^2 + 1)(s^2 - s)} \right)$$

$$Y(s) = e^{-(\pi/4)s} \left(\frac{s}{(s^2 + 1)s(s - 1)} \right)$$

...Partial Fractions Not Shown...

$$Y(s) = e^{-(\pi/4)s} \left(\frac{-\frac{1}{2}s - \frac{1}{2}}{s^2 + 1} + \frac{\frac{1}{2}}{s - 1} \right)$$

$$Y(s) = -\frac{1}{2} e^{-(\pi/4)s} \left(\frac{s + 1}{s^2 + 1} - \frac{1}{s - 1} \right)$$

$$Y(s) = -\frac{1}{2} e^{-(\pi/4)s} \left(\frac{s}{s^2 + 1} + \frac{1}{s^2 + 1} - \frac{1}{s - 1} \right)$$

Yields: $\cos(t) + \sin(t) - e^t$

$$y(t) = -\frac{1}{2} u_{\pi/4}(t) \left(\cos(t - \pi/4) + \sin(t - \pi/4) - e^{(t - \pi/4)} \right)$$