MATH 246: Chapter 3 section 1 Intro to First-Order Systems
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1. **Introduction:**

   All that we've studied so far are DEs involving a single function y depending on a single variable t. In the real world things can get far more complicated. As a classic example consider a predator-prey situation. The rate of growth of the prey depends on both the number of prey and the number of predators, as well as time, and similarly for the rate of growth of the predators.

   What we'll study now are systems of first-order DEs. In the most basic case we'll have two functions $x_1(t)$ and $x_2(t)$ in which their derivatives depend on both functions and maybe on other functions of t.

   Example:

   $$x_1' = 2x_1 - 3x_2 + \cos t$$
   $$x_2' = x_1 + 4x_2 - 2t$$

   The goal here would be to find a solution, meaning a pair of functions $x_1(t)$ and $x_2(t)$, which satisfy the system.

   We can also have more functions and equations.

   Example:

   $$x_1' = 2x_1 - 3x_2 - x_3 + \cos t$$
   $$x_2' = x_1 + 4x_2 - 2t$$
   $$x_3' = -x_1 + 20x_2 - 7x_3 + e^t$$

   The goal here would be to find a solution, meaning three functions $x_1(t)$, $x_2(t)$ and $x_3(t)$, which satisfy the system.

   In addition we could have an initial value, which would mean an initial value for each of the functions. So the example above could also have $x_1(0) = -2$, $x_2(0) = 3$ and $x_3(0) = 0$.

2. **Recasting Single Higher-Order:**

(a) Just the DE part:

Single higher-order DEs can be rewritten as systems of first-order DEs. This may be useful as we go on to develop methods of solving systems. The general idea for an nth order DE will be to rewrite it as a system of n first-order DEs. The method is easy - we assign:

$$x_1 = y$$
$$x_2 = y'$$
$$x_3 = y''$$
$$\vdots = \vdots$$
$$x_{n-1} = y^{(n-2)}$$
$$x_n = y^{(n-1)}$$

The first $n-1$ then give us:

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = x_4$$
$$\vdots = \vdots$$
$$x_{n-1}' = x_n$$

We get one more $x_n' = \ldots$ from the DE because $x_n' = y^{(n)}$, which we can find, and replacing all the other derivatives by their respective x_i.

Example: Consider $y''' - 2y'' + ty' - e^t y = \sin t$. We assign and get:

$$x_1 = y$$
$$x_2 = y'$$
$$x_3 = y''$$

The first two of these give us:

$$x_1' = x_2$$
$$x_2' = x_3$$

The third $x_3' = \ldots$ comes from the DE and is:

$$x_3' = y''' = 2y'' - ty' + e^t y + \sin t = 2x_3 - tx_2 + e^t x_1 + \sin t$$

All together we have the system:

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = 2x_3 - tx_2 + e^t x_1 + \sin t$$

(b) With Initial Values:

Initial values are easy to recast. Since we know $y(t_I), y'(t_I), ..., y^{(n-1)}(t_I)$ these just become $x_1(t_I), x_2(t_I), ..., x_n(t_I)$.

Example: In the previous example if we knew that $y(0) = 3$, $y'(0) = 2$ and $y''(0) = 0$ then we would get $x_1(0) = 3$, $x_2(0) = 2$ and $x_3(0) = 0$.

3. **Tank Problems:**

A classic example of these are tank problems. Imagine two tanks containing salt water. Water is being pumped into and out of these tanks in a variety of ways. For example in the following scenario there are two tanks. The one on the left contains 100G and the one on the right 200G. These quantities do not change in this example because the gallons into each equals the gallons out.
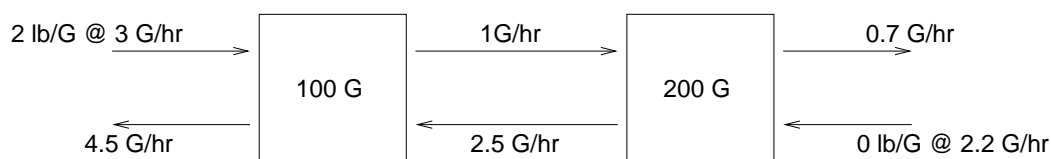
The quantity 2 lb/G 3 G/hr indicates that salt water with that density is flowing into the left tank at that rate, and the quantity 0 lb/G 2.2 G/hr indicates the same for the right tank.

The other quantities do not have densities becase they are assumed to be mixtures from the tank. More on this later.

Imagine now that x_1 is the amount of salt in the left tank and x_2 is the amount of salt in the right tank, each at time t. Then the density of salt in the left tank is $\frac{x_1}{100}$ and in the right tank is $\frac{x_2}{200}$. The entire scenario is then modeled by the system:

$$x'_1 = [\text{Rate of Salt In}] - [\text{Rate of Salt Out}]$$
$$x'_2 = [\text{Rate of Salt In}] - [\text{Rate of Salt Out}]$$

which is:

$$x'_1 = +(2)(3) + (x_2/200)(2.5) - (x_1/100)(4.5) - (x_1/100)(1)$$
$$x'_2 = +(x_1/100)(1) + (0)(2.2) - (x_2/200)(0.7) - (x_2/200)(2.5)$$

This simplifies to:

$$x'_1 = -0.055x_1 + 0.0125x_2 + 6$$
$$x'_2 = 0.01x_1 - 0.0475x_2$$