MATH 246: Chapter 3 section 2 Basic Theory
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1. **Notation for Systems of First Order Linear DEs**

   Using matrix notation, a system like this:

$$x_1' = 3x_1 + 2tx_2 + t$$
$$x_2' = t^2 x_1 + 3x_2$$

   can be rewritten like this:
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 2t \\ t^2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t \\ 0 \end{bmatrix}$$

   or even further as:
$$\bar{x}' = \begin{bmatrix} 3 & 2t \\ t^2 & 3 \end{bmatrix} \bar{x} + \begin{bmatrix} t \\ 0 \end{bmatrix}$$

   The advantage is then that a solution to the system is now a single $\bar{x}$ rather than two function x_1 and x_2.

   Whenever we have
$$\bar{x}' = A(t)\bar{x} + \bar{f}(t)$$
   then a homogeneous system has $\bar{f}(t) = \bar{0}$ and the system has constant coefficients then this means the matrix A is all constants.

   An initial value can then be written as $\bar{x}(t_I) = \bar{x}_I$, so for example if we had the initial value $x_1(0) = 3$, $x_2(0) = -2$ we could combine these as $\bar{x}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

   Example: Rewrite the IVP:

$$x_1' = tx_1 + x_2 + t \qquad x_1(1) = 4$$
$$x_2' = 5x_1 - 7x_2 \qquad x_2(1) = -1$$

   in matrix/vector form.
   Solution: It's just:

$$\bar{x}' = \begin{bmatrix} t & 1 \\ 5 & -7 \end{bmatrix} \bar{x} + \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ with } \bar{x}(1) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

2. **Theory for Homogeneous - Fundamental Sets**

(a) Note: In what follows I've written $n = 2$ to mean that I'm giving a specific example that generalizes. You could substitute $n = 3, 4, ...$ and the theory would still be good. In cases where it's not clear what happens for 3,4,... I've said more.

(b) Recall: A single solution to a system of $n = 2$ DEs involves a single $\bar{x}$. This single $\bar{x}$ really represents $n = 2$ functions, x_1 and x_2 (more if $n \geq 3$), but it's easier to understand as just one $\bar{x}$.

Example: The system $\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$ has solution $\bar{x} = \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix}$

(c) Fundamental Sets: A homogeneous system of $n = 2$ DEs has a fundamental set consisting of $n = 2$ solutions $\bar{x}_1$ and $\bar{x}_2$ (more if $n \geq 3$) The general solution to the system then consists of all linear combination of those $n = 2$ solutions. A fundamental set has nonzero Wronskian where

$$W[\bar{x}_1, \bar{x}_2] = \det[\bar{x}_1 \;\; \bar{x}_2]$$

That determinant is just found by dumping the vectors $\bar{x}_1$ and $\bar{x}_2$ together in in a matrix and going from there.

Example 1: The system $\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$ has solutions $\bar{x}_1 = \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix}$ and $\bar{x}_2 = \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}$. These form a fundamental pair because $W[\bar{x}_1, \bar{x}_2] = \det \begin{bmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{bmatrix} = -2e^{6t} \neq 0$. Consequently the general solution to the system is $\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2 = c_1 \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix} + c_2 \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}$.

Example 2: The system $\bar{x}' = \begin{bmatrix} t^2 & 2t - t^4 \\ 1 & -t^2 \end{bmatrix} \bar{x}$ has solutions $\bar{x}_1 = \begin{bmatrix} 1 + t^3 \\ t \end{bmatrix}$ and $\bar{x}_2 = \begin{bmatrix} t^2 \\ 1 \end{bmatrix}$. These form a fundamental pair because $W[\bar{x}_1, \bar{x}_2] = \det \begin{bmatrix} 1 + t^3 & t^2 \\ t & 1 \end{bmatrix} = 1 \neq 0$. Consequently the general solution to the system is $\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2 = c_1 \begin{bmatrix} 1 + t^3 \\ t \end{bmatrix} + c_2 \begin{bmatrix} t^2 \\ 1 \end{bmatrix}$.

3. **A Bit More Notation and Such**

(a) The Fundamental Matrix:
The fundamental set is often put together in a matrix by lining up the columns and called the *fundamental matrix*. This is usually denoted Ψ or $\Psi(t)$.

Example 1 and 2 Again:
In example 1 we have: $\Psi(t) = \begin{bmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{bmatrix}$
In example 2 we have: $\Psi(t) = \begin{bmatrix} 1+t^3 & t^2 \\ t & 1 \end{bmatrix}$.

(b) The Natural Fundamental Solutions and Natural Fundamental Matrix for a given time:
If we solve the two initial value problems:

$$\bar{x}' = A\bar{x} \text{ with } \bar{x}(t_I) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\bar{x}' = A\bar{x} \text{ with } \bar{x}(t_I) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

we get what are known together as the *natural fundamental set associated to* t_I and if we put these together in a matrix we get the *natural fundamental matrix associated to* t_I which is denoted Φ. There is only one of these for a given t_I.
This matrix Φ is very handy because if we are trying to solve an initial value problem with initial value $\bar{x}(t_I) = \bar{x}_I$ then the solution to the initial value problem is $\bar{x} = \Phi\bar{x}_I$.
Even better if for some reason we already have any fundamental matrix Ψ then we can find Φ for a given t_I via $\Phi = \Psi(t)\Psi(t_I)^{-1}$.

Example 1 Again:
The system:

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$$

we saw has fundamental matrix:

$$\Psi(t) = \begin{bmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{bmatrix}$$

Suppose we wish to solve the IVP with $\bar{x}(0) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$.
We first find Φ by doing the following:

$$\begin{aligned} \Phi &= \Psi(t)\Psi(0)^{-1} \\ &= \begin{bmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{bmatrix} \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{5t}+e^{t} & e^{5t}-e^{t} \\ e^{5t}-e^{t} & e^{5t}+e^{t} \end{bmatrix} \end{aligned}$$

Then the solution to the IVP is given by:

$$\bar{x} = \Phi\bar{x}_I = \frac{1}{2} \begin{bmatrix} e^{5t}+e^{t} & e^{5t}-e^{t} \\ e^{5t}-e^{t} & e^{5t}+e^{t} \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} e^{5t}+3e^{t} \\ e^{5t}-3e^{t} \end{bmatrix}$$

Example:
The system:

$$\bar{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \bar{x}$$

has fundamental matrix:

$$\Psi(t) = \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix}$$

Suppose we wish to solve the IVP with $\bar{x}(0) = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$.
We first find Φ by doing the following:

$$\begin{aligned} \Phi &= \Psi(t)\Psi(0)^{-1} \\ &= \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} \frac{1}{-1} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \end{aligned}$$

Then the solution to the IVP is given by:

$$\bar{x} = \Phi \bar{x}_I = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 2\cos x + 7\sin x \\ -2\sin x + 7\cos x \end{bmatrix}$$

(c) **Nonhomogenous Systems**
The general idea here will be exactly the same as before. We will find the fundamental set for the homogeneous system and just one particular solution $\bar{x}_p$ for the nonhomogeneous system. We will then add $\bar{x}_p$ plus all linear combinations of the fundamental set. More on this later, nothing to obsess over yet.