MATH 246: Chapter 3 section 3 Matrix and Vector Essentials
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1. **Introduction:**

   It's far easier to manage systems of differential equations when we can rephrase them in the language of matrices and vectors. To that end, here are the essentials.

2. **Basic Definitions:**

   (a) A *matrix* is a rectangular array of numbers. It has size $m \times n$ if there are m rows and n columns. Matrices are typically denoted using capital letters:

   Example: Here is a 3×4 matrix:

   $$A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 5 & 0 & 17 \\ 2 & 2 & -7 & 3 \end{bmatrix}$$

   (b) Most of the matrices in this class will be *square*, meaning they have the same number of rows and columns. Mostly we'll deal with 2×2 and 3×3 matrics.

   (c) The *identity matrix* I_n is the square $n \times n$ matrix with 1s on the *main diagonal* (upper-left to lower right) and 0's elsewhere. When the size is clear from context we just write I.

   Example:

   $$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   (d) The *zero matrix* is matrix of all zeros.

   (e) A *vector* is a matrix which is a single column. Vectors are usually denoted in lower-case with a bar over the letter.

   Example: $\bar{a} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

3. **Combining Matrices and Vectors:**

(a) We can add matrices and vectors by adding matching entries provided they both have the same size.

Example: For example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 1-1 & 2+0 \\ 3+6 & 4+2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 9 & 8 \end{bmatrix}$$

(b) We can multiple an $n \times n$ matrix A by a vector $\bar{x}$ with n entries to get a new vector with n entries. We do this by multiplying each row of the matrix by the vector (element by element and add). This is easier to see:

Example: We have:

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 7 \\ 8 & -1 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} (1)(5) + (2)(3) + (-3)(2) \\ (0)(5) + (4)(3) + (7)(2) \\ (8)(5) + (-1)(3) + (5)(2) \end{bmatrix} = \begin{bmatrix} 5 \\ 26 \\ 47 \end{bmatrix}$$

(c) We can multiple an $n \times n$ matrix by another $n \times n$ matrix by multiplying the first matrix by each of the columns in the second matrix as if it were just a vector, then taking these new vectors an putting them together in a new matrix.

Example: Here it is with lots of brackets to help you see what's going on:

$$\begin{aligned} \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 5 & 9 \end{bmatrix} &= \left[\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \end{bmatrix} \right] \\ &= \left[\begin{bmatrix} (2)(-1) + (1)(5) \\ (4)(-1) + (3)(5) \end{bmatrix} \begin{bmatrix} (2)(3) + (1)(9) \\ (4)(3) + (3)(9) \end{bmatrix} \right] \\ &= \begin{bmatrix} 3 & 15 \\ 9 & 39 \end{bmatrix} \end{aligned}$$

(d) It's almost always the case that for matrices A and B that $AB \neq BA$.

(e) The identity matrix acts like the number 1 in that for any matrix A we have:

$$AI = IA = A$$

4. **Inverses:**

   (a) The *inverse* of an $n \times n$ matrix A is another matrix denoted A^{-1} such that $AA^{-1} = A^{-1}A = I$. It's like a "reciprocal" for matrics.

   (b) For the 2×2 size there is a formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example: For example:

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1} = \frac{1}{(1)(-2) - (3)(2)} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \frac{1}{-8} \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1/4 & 3/8 \\ -1/8 & -1/4 \end{bmatrix}$$

   (c) Properties include:
   - $\left(A^{-1}\right)^{-1} = A$
   - $(AB)^{-1} = B^{-1}A^{-1}$
   - $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$

5. **Determinants:**

   (a) The *determinant* of a matrix, denoted $\det A$, is a number calculated from the matrix. We've seen this for 2×2 and 3×3 matrices.

   (b) Properties include:
   - A matrix A has an inverse if and only if $\det A \neq 0$.
   - For a 2×2 case $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.
   - We have $\det\left(A^{-1}\right) = 1/(\det A)$.

6. **Minutia:**

   (a) The *transpose* of an $n \times n$ matrix A, denoted A^T, is found by reflecting the matrix in its main diagonal.

Example: We have:

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 7 \\ 8 & -1 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 8 \\ 2 & 4 & -1 \\ -3 & 7 & 5 \end{bmatrix}$$

   (b) A matrix may have complex numbers in it, in which case its *comples conjugate* denoted either $\bar{A}$ or A^c, is found by taking the complex conjugate of each entry.

Example: We have:

$$\begin{bmatrix} 1 - 2i & 5 \\ 5 + i & 7 + 8i \end{bmatrix}^C = \begin{bmatrix} 1 + 2i & 5 \\ 5 - i & 7 - 8i \end{bmatrix}$$