MATH 246: Chapter 3 sections 3 and 4
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1. **A Funny Suggestion:**

   Recall the really easy solution to a single first-order linear homogeneous differential equation with constant coefficients:

   $$y' = ay \quad \Longrightarrow \quad y = e^{at}$$

   It would be rather hilarious if the same were true of systems, meaning if we have a system of first-order linear homogeneous differential equations with constant coefficients:

   $$\bar{x}' = A\bar{x} \quad \Longrightarrow \quad \underbrace{y = e^{tA}}_{LOL}$$

   The issue is that it's not clear to us that e^{tA} can make any sense at all when A is an $n \times n$ matrix.

   However it can and the reason is as follows - recall that the Taylor Expansion of e^x tells us:

   $$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...$$

   We can define a matrix exponential similarly:

   $$e^{tA} = 1 + (tA) + \frac{1}{2!}(tA)^2 + \frac{1}{3!}(tA)^3 + ... = 1 + ta + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + ...$$

   This makes sense, mathematically, and in fact it converges, but we won't show that here.

   Interestingly using this definition the matrix e^{tA} does give us all the solutions to $\bar{x}' = A\bar{x}$ in that the columns of e^{tA} form a fundamental set of solutions but it's sort of useless because knowing the series expansion above doesn't help us actually calculate it.

   So the question then becomes - can we find e^{tA} some other way?

2. **Eigenstuff:**

If we have a matrix, the determinant is the most important number associated to it. After the determinant the next most important items are eigenvalues and eigenvectors.

(a) If A is an $n \times n$ matrix, an *eigenvalue* of A is a number λ with the property that there is some $\bar{v} \neq \bar{0}$ such that $A\bar{v} = \lambda\bar{v}$. The vector $\bar{v}$ is then an *eigenvector* associated to λ and we say that $(\lambda, \bar{v})$ is an *eigenpair* of A.

Example: Observe that:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so we would say that $\lambda = 3$ is an eigenvalue, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector and $\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ is an eigenpair for the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

(b) Any nonzero multiple of an eigenvector is also an eigenvector, so in the above example $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 17 \\ 17 \end{bmatrix}$ and $\begin{bmatrix} -7 \\ -7 \end{bmatrix}$ are all eigenvectors for the same eigenvalue.

(c) If we have a complex eigenpair $(\lambda, \bar{v})$ then $(\bar{\lambda}, \bar{\bar{v}})$ is also an eigenpair.

(d) Finding eigenpairs will be essential to solving systems of linear homogeneous differential equations with constant coefficients. The process to finding these is theorically straightfoward and a bit calculationally awkward. We'll focus on the 2×2 case and do higher cases via Matlab.

(e) Given an $n \times n$ matrix A, the *characteristic polynomial* of A is defined as

$$p(z) = \det(zI - A)$$

The eigenvalues of A are the roots of this characteristic polynomial.

Example: To find the eigenvalues of

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

we find

$$\begin{aligned} p(z) &= \det(zI - A) \\ &= \det\left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\right) \\ &= \det \begin{bmatrix} z-3 & -2 \\ -2 & z-3 \end{bmatrix} \\ &= (z-3)(z-3) - 4 \\ &= z^2 - 6z + 5 \\ &= (z-5)(z-1) \end{aligned}$$

The eigenvalues are then the roots so $\lambda_1 = 5$ and $\lambda_2 = 1$.

Example: To find the eigenvalues of

$$A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$$

we find

$$\begin{aligned} p(z) &= \det(zI - A) \\ &= \det\left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \right) \\ &= \det\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \right) \\ &= \det \begin{bmatrix} z-4 & 1 \\ -1 & z-2 \end{bmatrix} \\ &= (z-4)(z-2) + 1 \\ &= z^2 - 6z + 9 \\ &= (z-3)^2 \end{aligned}$$

The only eigenvalue is the root $\lambda = 3$. However this multiplicity counts, so we can think $\lambda_1 = 3$ and $\lambda_2 = 3$.

Example: To find the eigenvalues of

$$A = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$$

we find

$$\begin{aligned} p(z) &= \det(zI - A) \\ &= \det\left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} \right) \\ &= \det\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} \right) \\ &= \det \begin{bmatrix} z-3 & -2 \\ 2 & z-3 \end{bmatrix} \\ &= (z-3)(z-3) + 4 \\ &= z^2 - 6z + 13 \end{aligned}$$

This does not factor so we use the quadratic formula:

$$z = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(13)}}{2} = 3 \pm 2i$$

The eigenvalues are then $\lambda_1 = 3 + 2i$ and $\lambda_2 = 3 - 2i$.

(f) Once we find the eigenvalues we take each eigenvalue and solve the matrix equation $A\bar{v} = \lambda\bar{v}$, or $A\bar{v} - \lambda\bar{v} = \bar{0}$, or $(A - \lambda I)\bar{v} = \bar{0}$. This can be fairly intensive for large cases. For the 2×2 case there is a trick, though, which is really useful:

For λ_1 choose any nonzero column of $A - \lambda_2 I$.
For λ_2 choose any nonzero column of $A - \lambda_1 I$.

Example: We saw that the eigenvalues for $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ are $\lambda_1 = 5$ and $\lambda_2 = 1$. Then:
For $\lambda_1 = 5$ choose any nonzero column of $A - \lambda_2 I = A - 1I = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$
so $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ will do. However since any multiple of this works, we can pick the nicer $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
For $\lambda_2 = 1$ choose any nonzero column of $A - \lambda_1 I = A - 5I = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$
so $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ will do. However since any multiple of this works, we can pick the nicer $\bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
We thus have eigenpairs $\left(5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ and $\left(1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

Example: We saw that the eigenvalue for $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$ is $\lambda_1 = \lambda_2 = 3$. Then:
For $\lambda_1 = 3$ choose any nonzero column of $A - \lambda_2 I = A - 3I = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$
so $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ will do.
Notice that $\lambda_2 = \lambda_1$ so we get nothing new.
We thus have the single eigenpair $\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

Example: We saw that the eigenvalues for $A = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$ are $\lambda_1 = 3 + 2i$ and $\lambda_2 = 3 - 2i$. Then:
For $\lambda_1 = 3 + 2i$ choose any nonzero column of $A - \lambda_2 I = A - (3 + 2i)I = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} - \begin{bmatrix} 3 + 2i & 0 \\ 0 & 3 + 2i \end{bmatrix} = \begin{bmatrix} 2i & 2 \\ -2 & -2i \end{bmatrix}$ so $\begin{bmatrix} 2i \\ -2 \end{bmatrix}$ will do. However since any multiple of this works, we can pick the nicer $\bar{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$.
We know from earlier that for $\lambda_2 = 3 - 2i$ we can use the conjugate so $\bar{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.
We thus have eigenpairs $\left(3 + 2i, \begin{bmatrix} 1 \\ i \end{bmatrix}\right)$ and $\left(3 - 2i, \begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$.

3. **Using Eigenpairs to Construct Solutions:**

If we go back to $\bar{x}' = A\bar{x}$ observe that if $(\lambda, \bar{v})$ is an eigenpair then it turns out that $\bar{x} = e^{\lambda t}\bar{v}$ is a solution:

$$\bar{x}' = \frac{d}{dt}(e^{\lambda t}\bar{v}) = e^{\lambda t}\lambda\bar{v} = e^{\lambda t}A\bar{v} = Ae^{\lambda t}\bar{v} = A\bar{x}$$

This tells us that we can construct a fundamental set using the eigenpairs. However there are some stumbling blocks.

What follows is all for 2×2:

(a) If we have two real eigenpairs then we get two solutions and they will form a fundamental set.

Example: The system

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$$

has a matrix with eigenpairs $\left(5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ and $\left(1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$. Therefore we have two solutions

$$\bar{x}_1 = e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \bar{x}_2 = e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so the general solution is:

$$\bar{x} = c_1e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(b) If we have just one eigenpair (with multiplicity) then the situation is trickier. It turns out that if $(\lambda, \bar{v})$ is the single eigenpair of a 2×2 matrix then a second solution can be obtained by letting $\bar{w}$ be some nonzero vector which is not a multiple of $\bar{v}$ and then

$$x_2 = e^{\lambda t}\bar{w} + te^{\lambda t}(A - \lambda I)\bar{w}$$

will be another solution.

Example:

$$\bar{x}' = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \bar{x}$$

has matrix with eigenpair $\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. This gives us one solution $\bar{x}_1 = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

To find another choose $\bar{w}$ to be any non-multiple of $\bar{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, for example $\bar{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then a second solution is:

$$x_2 = e^{\lambda t}\bar{w} + te^{\lambda t}(A - \lambda I)\bar{w} = e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + te^{3t}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + te^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{3t}\begin{bmatrix} 1+t \\ t \end{bmatrix}$$

so the general solution is

$$\bar{x} = c_1e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 1+t \\ t \end{bmatrix}$$

(c) If we have two complex conjugate eigenpairs then we do get two solutions but they are not real solutions. We've seen this issue before.

Here is the long way to think about it and then the short way. The long way is just to make sure we understand why the short way works.

Example:

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} \bar{x}$$

has a matrix with eigenpairs $\left(3+2i, \begin{bmatrix} 1 \\ i \end{bmatrix}\right)$ and $\left(3-2i, \begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$.

Long Way: The first gives us the solution:

$$\begin{aligned}
\bar{x} &= e^{(3+2i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} \\
&= \left(e^{3t}\cos(2t) + ie^{3t}\sin(2t)\right) \begin{bmatrix} 1 \\ i \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}\cos(2t) + ie^{3t}\sin(2t) \\ ie^{3t}\cos(2t) - e^{3t}\sin(2t) \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}\cos(2t) + ie^{3t}\sin(2t) \\ -e^{3t}\sin(2t) + ie^{3t}\cos(2t) \end{bmatrix}
\end{aligned}$$

The second gives us the solution:

$$\begin{aligned}
\bar{x} &= e^{(3-2i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix} \\
&= \left(e^{3t}\cos(-2t) + ie^{3t}\sin(-2t)\right) \begin{bmatrix} 1 \\ -i \end{bmatrix} \\
&= \left(e^{3t}\cos(2t) - ie^{3t}\sin(2t)\right) \begin{bmatrix} 1 \\ -i \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}\cos(2t) - ie^{3t}\sin(2t) \\ -ie^{3t}\cos(2t) - e^{3t}\sin(2t) \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}\cos(2t) - ie^{3t}\sin(2t) \\ -e^{3t}\sin(2t) - ie^{3t}\cos(2t) \end{bmatrix}
\end{aligned}$$

Since linear combinations of solutions are solutions if we take half of the sum of these we get the solution:

$$\bar{x}_1 = \frac{1}{2}\left(\begin{bmatrix} e^{3t}\cos(2t) + ie^{3t}\sin(2t) \\ -e^{3t}\sin(2t) + ie^{3t}\cos(2t) \end{bmatrix} + \begin{bmatrix} e^{3t}\cos(2t) - ie^{3t}\sin(2t) \\ -e^{3t}\sin(2t) - ie^{3t}\cos(2t) \end{bmatrix}\right) = e^{3t}\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}$$

and if we take $\frac{1}{2i}$ times the difference we get the solution:

$$\bar{x}_1 = \frac{1}{2i}\left(\begin{bmatrix} e^{3t}\cos(2t) + ie^{3t}\sin(2t) \\ -e^{3t}\sin(2t) + ie^{3t}\cos(2t) \end{bmatrix} - \begin{bmatrix} e^{3t}\cos(2t) - ie^{3t}\sin(2t) \\ -e^{3t}\sin(2t) - ie^{3t}\cos(2t) \end{bmatrix}\right) = e^{3t}\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

Short Way: Look back at our very first $\bar{x}$ and break it into real and imaginary parts:

$$\bar{x} = \begin{bmatrix} e^{3t}\cos(2t) + ie^{3t}\sin(2t) \\ -e^{3t}\sin(2t) + ie^{3t}\cos(2t) \end{bmatrix} = e^{3t}\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + ie^{3t}\begin{bmatrix} \sin(2t) \\ \cos(2t \end{bmatrix}$$

The two solutions are just the real and imaginary parts:

$$\bar{x} = \underbrace{e^{3t}\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}}_{\bar{x}_1} + i \underbrace{e^{3t}\begin{bmatrix} \sin(2t) \\ \cos(2t \end{bmatrix}}_{\bar{x}_2}$$

so the general solution is:

$$\bar{x} = c_1\begin{bmatrix} e^{3t}\cos(2t) \\ -e^{3t}\sin(2t) \end{bmatrix} + c_2\begin{bmatrix} e^{et}\sin(2t) \\ e^{3t}\cos(2t) \end{bmatrix}$$

(d) **An Initial Value Problem:** Since we haven't done one from start to finish, here is an initial value problem:

Example: Solve

$$\bar{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \text{ with } \bar{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

i. Find the eigenvalues:

$$p(z) = \det \begin{bmatrix} z-5 & -1 \\ 3 & z-1 \end{bmatrix} = (z-5)(z-1) - (-1)(3) = z^2 - 6z + 8 = (z-2)(z-4)$$

So the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 4$.

ii. Find the eigenvectors:

For $\lambda_1 = 2$ choose a nonzero column of $A - \lambda_2 I = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$ so $\bar{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

For $\lambda_2 = 4$ choose a nonzero column of $A - \lambda_1 I = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$ so $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

iii. Write down the general solution:
We have

$$\bar{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

iv. Plug in the initial value and solve:

$$\bar{x}(0) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

so that

$$\begin{aligned} c_1 + c_2 &= 2 \\ 3c_1 + c_2 &= 3 \end{aligned}$$

So then $c_1 = \frac{1}{2}$ and $c_2 = \frac{3}{2}$.

v. Write down the answer:

$$\bar{x} = \frac{1}{2} e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{3}{2} e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(e) **Back to the Exponential:**

We never actually came back to e^{tA}. It turns out that if $\Psi(t)$ is any fundamental matrix for $\bar{x}' = A\bar{x}$ then

$$e^{tA} = \Psi(t)\Psi(0)^{-1}$$

In other words e^{tA} is the natural fundamental matrix associated to $t_I = 0$.

Example: Consider

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

To find e^{tA} we note that the system $\bar{x}' = A\bar{x}$ has fundamental matrix

$$\Psi(t) = \begin{bmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{bmatrix}$$

and so

$$\begin{aligned} e^{tA} &= \begin{bmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{bmatrix} \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{5t} + e^{t} & e^{5t} - e^{t} \\ e^{5t} - e^{t} & e^{5t} + e^{t} \end{bmatrix} \end{aligned}$$