MATH 246: Chapter 3 Section 6
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1. **Introduction:**

   The goal of this section is to understand what the solutions of the system:

   $$\bar{x}' = A\bar{x}$$

   look like graphically when we are in the $n = 2$ case.

   To make this a little clearer instead of thinking of $\bar{x}'$ as $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ we'll think of $\bar{x}'$ as $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ because this way a solution can be thought of as a curve which moves around in the xy-plane as a function of t.

   In addition we'll think of the matrix A as:

   $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

   What we'll do first is go through one example thoroughly and then the rest will be categorized without too much explanation.

2. **Broad Strokes:**

   The general idea is that the solutions can always be graphed using just the eigenvalues and sometimes (but not always) the eigenvectors. The solutions will not be perfect but they'll give us a lot of insight.

   Here the types of solutions have been broken down into five categories to make them easier to remember. Each category has subcategories.

   While this seems like a lot there are many similarities and you'll find that patters repeat over and over and make a lot of sense, so it's really not that terrible!

3. **First Example:**

Consider the system:

$$\bar{x}' = \begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix} \bar{x}$$

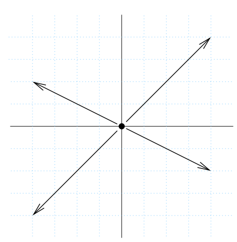
The eigenpairs of $A = \begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix}$ are $\left(3, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$ and $\left(9, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. The general solution is then

$$\bar{x} = C_1 e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let's analyze a few solutions:

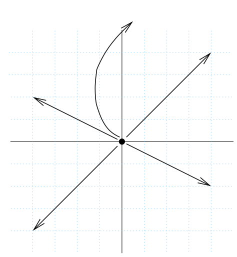
- If $C_1 = 0$ and $C_2 = 0$ then we get $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ which is a constant solution which sits at the origin for all t.
- If $C_1 = 0$ and $C_2 = 1$ then we get $\bar{x} = e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{9t} \\ e^{9t} \end{bmatrix}$ Notice that $x(t) = e^{9t}$ and $y = e^{9t}$ are always positive and equal. As $t \to \infty$ this point moves away from the origin and as $t \to -\infty$ this point moves toward but never touches (slows down as it goes) the origin.
- If $C_1 = 0$ and $C_2 = -1$ we get a similar thing, the only difference being that both $x(t)$ and $y(t)$ are negative.
- If $C_1 = 1$ and $C_2 = 0$ then we get $\bar{x} = e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2e^{3t} \\ e^{3t} \end{bmatrix}$. This solution always has $x = -2y$ but otherwise has the same behavior.
- If $C_1 = -1$ and $C_2 = 0$ we get the opposite of the previous.

All together we get the following five solutions:



One more solution:

- If $C_1 = 1$ and $C_2 = 1$ then we get $\bar{x} = e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For large negative t the e^{9t} is closer to zero than the e^{3t} and so the function behaves like $e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. On the other hand for large positive t the e^{3t} still exists and contributes but the e^{9t} is much more significant and so the function turns out to be approaching parallel to $e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The result is:

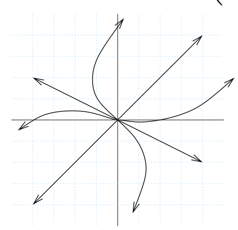


4. **Categories of Solutions:**

(a) The eigenvalues λ_1, λ_2 are both real, nonzero and different.

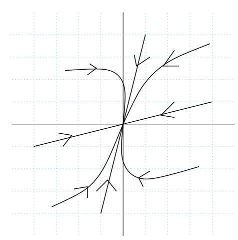
i. Both eigenvalues are positive: **Nodal Source - Unstable**
In this case there are four straight-line solutions moving away from the origin along the vectors $\pm v_1$ and $\pm v_2$. The other solutions move away from the origin too, however when they are close to the origin they are tangent to the eigenvector whose eigenvalue is closest to 0 and when they are far from the origin they are tangent to the eigenvector whose eigenvalue is furthest from 0.

Example 1: If $A = \begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix}$ then the epairs are $\left(3, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right), \left(9, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. Picture:



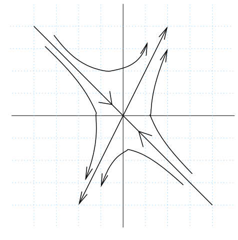
ii. Both eigenvalues are negative: **Nodal Sink - Stable**
In this case there are four straight-line solutions moving toward the origin along the vectors $\pm v_1$ and $\pm v_2$. The other solutions move toward from the origin too, however when they are close to the origin they are tangent to the eigenvector whose eigenvalue is closest to 0 and when they are far from the origin they are tangent to the eigenvector whose eigenvalue is furthest from 0.

Example 2: If $A = \begin{bmatrix} -46 & 4 \\ -4 & -29 \end{bmatrix}$ then the epairs are $\left(-45, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right), \left(-30, \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$.
Picture:



iii. One eigenvalue is positive and the other is negative: **Saddle - Unstable**
In this case there are four straight-line solutions. The two corresponding to the positive eigenvalue move away from the origin (along the eigenvector and its opposite) and the two corresponding to the negative eigenvalue move toward the origin (along the eigenvector and its opposite). The other solutions move toward the origin initially parallel to the straight-line solutions moving toward the origin but then curve and move away parallel to the other straight-line solution.

Example 3: If $A = \begin{bmatrix} -5 & 4 \\ 8 & -1 \end{bmatrix}$ then the epairs are $\left(-9, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right), \left(3, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$. Picture:

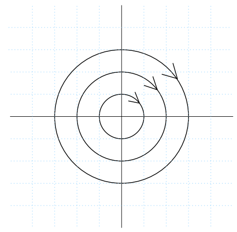


(b) The eigenvalues are complex conjugates.

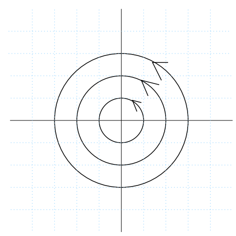
i. They have the form $0 \pm si$: **Circle - Stable**

In this case the solutions are circles around the origin. They are clockwise if $a_{12} > 0$ and counterclockwise if $a_{12} < 0$.

Example 4: If $A = \begin{bmatrix} -2 & 2 \\ -4 & 2 \end{bmatrix}$ then the evals are $\pm 2i$. Picture:

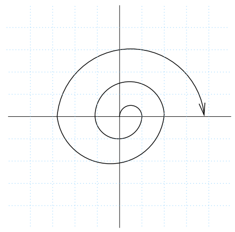


Example 5: If $A = \begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix}$ then the evals are $\pm 6i$. Picture:

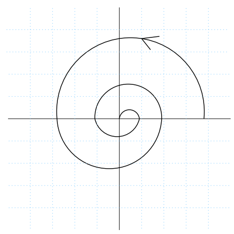


ii. They have the form $r \pm si$: **Spiral Source - Unstable (if out) or Sink - Stable (if in)**
In this case the solutions are spirals around the origin. They are clockwise if $a_{12} > 0$ and counterclockwise if $a_{12} < 0$ and they spiral in if $r < 0$ and out if $r > 0$.

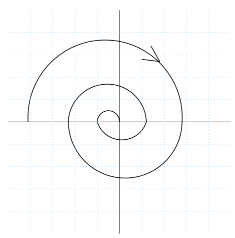
Example 6: If $A = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix}$ then the evals are $3 \pm 2i$. Picture:



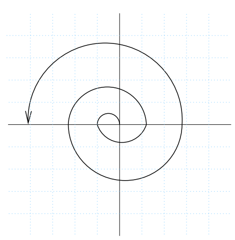
Example 7: If $A = \begin{bmatrix} -1 & -2 \\ 4 & -5 \end{bmatrix}$ then the evals are $-3 \pm 2i$. Picture:



Example 8: If $A = \begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix}$ then the evals are $-2 \pm 3i$. Picture:



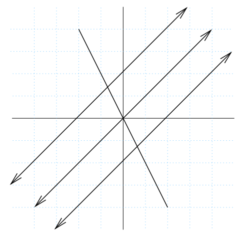
Example 9: If $A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$ then the evals are $2 \pm 6i$. Picture:



(c) One eigenvalue is 0, the other λ is real and not zero.

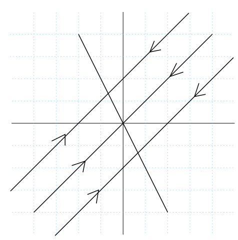
i. The other eigenvalue is positive: **Linear Source - Unstable**
In this case the line along the eigenvector whose eigenvalue is 0 is a line of stationary solutions - basically a bunch of points. The other solutions all move away from that line and are parallel to the eigenvector corresponding to λ.

Example 10: If $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$ then the epairs are $\left(0, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right), \left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. Picture:



ii. The other eigenvalue is negative: **Linear Sink - Stable**
In this case the line along the eigenvector whose eigenvalue is 0 is a line of stationary solutions - basically a bunch of points. The other solutions all move toward that line and are parallel to the eigenvector corresponding to λ.

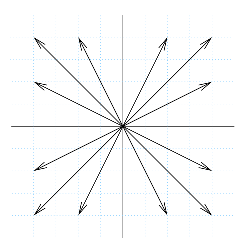
Example 11: If $A = \begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix}$ then the epairs are $\left(0, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right), \left(-3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. Picture:



(d) There is a single nonzero eigenvalue λ and A looks like $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$:

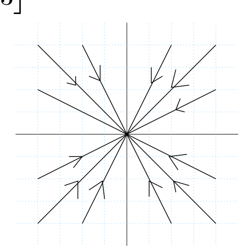
i. If the eigenvalue is positive: **Radial Source - Unstable**
In this case all the solutions are straight lines moving away from the origin.

Example 12: If $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ then the eval is 2. Picture:



ii. If the eigenvalue is negative: **Radial Sink - Stable**
In this case all the solutions are straight lines moving toward the origin.

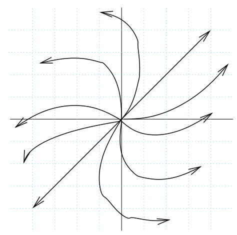
Example 13: If $A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$ then the eval is -3. Picture:



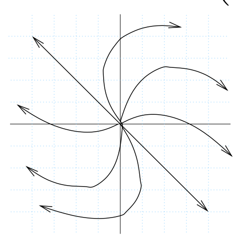
(e) There is a single nonzero eigenvalue λ and A does not look like that:

i. If the eigenvalue is positive: **Twist Source - Unstable**
In this case there are two straight-line solutions moving away from the origin along the eigenvector corresponding to λ. The other solutions are all curved solutions which move away from the origin in a clockwise direction if $a_{12} > 0$ and in a counterclockwise direction if $a_{12} < 0$.

Example 14: If $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$ then the epair is $\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. Picture:

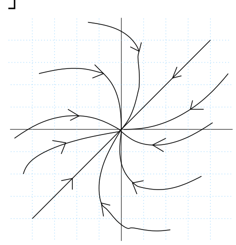


Example 15: If $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$ then the epair is $\left(3, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$. Picture:

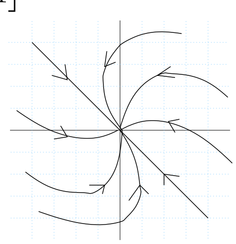


ii. If the eigenvalue is negative: **Twist Sink - Stable**
In this case there are two straight-line solutions moving toward the origin along the eigenvector corresponding to λ. The other solutions are all curved solutions which move toward the origin in a clockwise direction if $a_{12} > 0$ and in a counterclockwise direction if $a_{12} < 0$.

Example 16: If $A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$ then the epair is $\left(-2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. Picture:

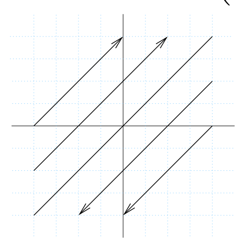


Example 17: If $A = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix}$ then the epair is $\left(-2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$. Picture:



iii. If the eigenvalue is zero: **Parallel Shear - Unstable**
In this case the line along the eigenvector whose eigenvalue is 0 is a line of stationary solutions. The other solutions are straight lines parallel to that one, "clockwise" if $a_{12} > 0$ and "counterclockwise" if $a_{12} < 0$.

Example 18: If $A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ then the epair is $\left(0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. Picture:



Example 19: If $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ then the epair is $\left(0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. Picture: