MATH 246: Chapter 3 Section 7
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1. **Introduction**

   The goal of this section is to look at a couple of more specific things related to systems of two differential equations. We're going to first throw out the requirement that the system is linear, meaning we can't assume it looks like $\bar{x}' = A\bar{x}$. Instead we'll think of these as:

$$x' = f(x, y)$$
$$y' = g(x, y)$$

   Example: One example would be something like:

$$x' = x^2 - y^2$$
$$y' = y + 2$$

2. **Stationary Solutions**

   The first and easiest solutions to look for are those that are constant, or stationary, meaning they do not move for all t, meaning x' and y' are both zero. We can find these by simply setting $f(x, y) = 0$ and $g(x, y) = 0$ and solving.

   Example: Find the stationary solutions to:

$$x' = x^2 - y^2$$
$$y' = y + 2$$

   We set $x^2 - y^2 = 0$ and $y + 2 = 0$. The latter gives us $y = -2$ and then the former gives us $x^2 - 4 = 0$ so $x = \pm 2$. Therefore there are two stationary solutions, $(2, -2)$ and $(-2, -2)$.

3. **Hamiltonian Systems**

(a) Definition of Hamiltonian

One very special type of system of differential equations is a Hamiltonian system. A Hamiltonian system is a system in which there is some function $H(x, y)$ such that:

$$\begin{aligned} \frac{dx}{dt} &= H_y(x, y) \\ \frac{dy}{dt} &= -H_x(x, y) \end{aligned}$$

The reason that these are nice is that for a Hamiltonian system we have:

$$\begin{aligned} -H_x(x, y)\frac{dx}{dt} &= H_y(x, y)\frac{dy}{dt} \\ H_x(x, y)\frac{dy}{dt} + H_y(x, y)\frac{dy}{dt} &= 0 \\ \frac{d}{dt}H(x, y) &= 0 \\ H(x, y) &= C \end{aligned}$$

For some/any constant C. This means that solutions to the system of differential equations are level curves for H.

Example: The system:

$$\begin{aligned} \frac{dx}{dt} &= 2y \\ \frac{dy}{dt} &= -2x \end{aligned}$$

is Hamiltonian with $H(x, y) = x^2 + y^2$. The solutions then satisfy $x^2 + y^2 = C$ and so they're circles.

Notice that we could also have seen this using methods from the previous section. Here $\bar{x}' = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \bar{x}$ so the eigenvalues for A are $0 \pm 2i$ and since $a_{12} > 0$ the solutions are (clockwise) circles.

(b) Is a System Hamiltonian and Finding $H(x, y)$.

It turns out that a system is Hamiltonian if $f_x + g_y = 0$ and if it is then we can find H using a process we used before with exact differential equations.

Example: Show the following system is Hamiltonian and find $H(x, y)$:

$$\begin{aligned} \frac{dx}{dt} &= x^2 + 2y \\ \frac{dy}{dt} &= -2xy \end{aligned}$$

First note that $f_x + g_y = 2x - 2x = 0$ so the system is Hamiltonian. We wish to find $H(x, y)$ with $H_y(x, y) = x^2 + 2y$ and $-H_x(x, y) = -2xy$. The latter tells us $H_x(x, y) = 2xy$ and so $H(x, y) = x^2y + g(y)$. From here $H_y(x, y) = x^2 + g'(y) = x^2 + 2y$ so $g'(y) = 2y$ and $g(y) = y^2 + C$. Then $H(x, y) = x^2y + y^2 + C$. Since we can choose any constant we let $H(x, y) = x^2y + y^2$. Thus the solutions, when plotted, satisfy the equation $x^2y + y^2 = C$, whatever this looks like!

(c) Deeper analysis of Hamiltonian systems - Graphing.

Hamiltonian systems can be analyzed further by looking at the Hessian at each stationary point:

$$\partial^2 H = \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix}$$

We'll only look at those for which the determinant of the Hessian is nonzero.

If $\det \partial^2 H < 0$ then the stationary point is a saddle.

If $\det \partial^2 H > 0$ then the stationary point is a circle.

Directions can be figured out by testing points.

Example: Consider the system

$$\frac{dx}{dt} = 4y - y^3$$
$$\frac{dy}{dt} = x$$

To find the stationary solutions we set $4y - y^3 = y(4 - y^2) = 0$ and $x = 0$. The former gives us $y = 0, \pm 2$ so there are three stationary solutions at $(0,0)$, $(0,2)$ and $(0,-2)$.

Noting that $H_y = 4y - y^3$ and $-H_x = x$, so $H_x = -x$, we get the Hessian:

$$\partial^2 H = \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 4 - 3y^2 \end{bmatrix}$$

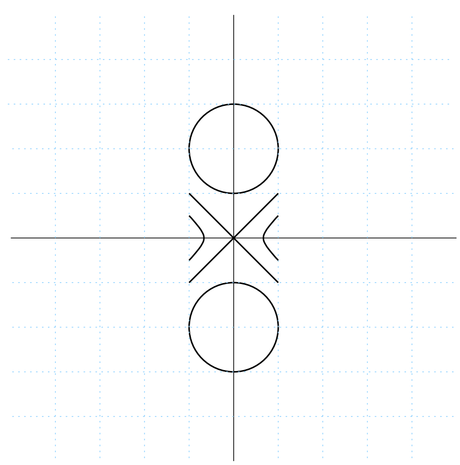
Then at each point:

$$\det \partial^2 H(0,0) = \det \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} = -4 \text{ so } (0,0) \text{ is a saddle}$$

$$\det \partial^2 H(0,2) = \det \begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix} = 8 \text{ so } (0,2) \text{ is a circle}$$

$$\det \partial^2 H(0,-2) = \det \begin{bmatrix} -1 & 0 \\ 0 & -8 \end{bmatrix} = 8 \text{ so } (0,-2) \text{ is a circle.}$$

A preliminary picture is:



Notice how the circles fit nicely with the saddle shape!

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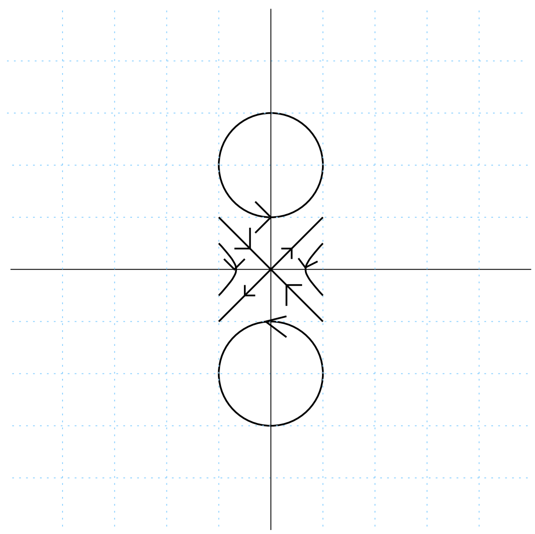
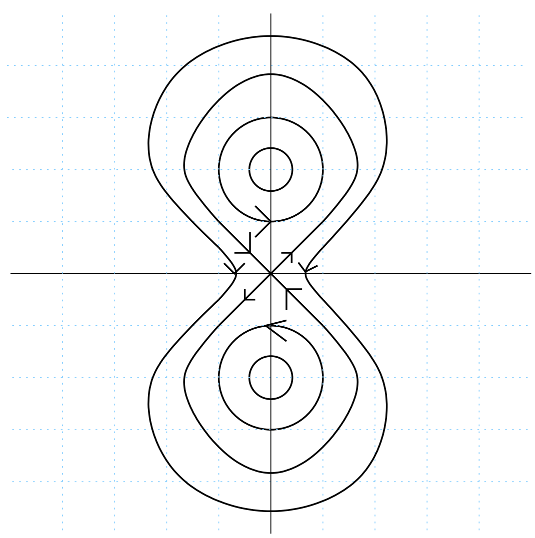
Example (Continued):
We need to know what direction everything goes in. Interestingly, for this picture, we can find everything out by testing one point in the system. At the point $(0,1)$ we have:

$$\frac{dx}{dt} = 3$$
$$\frac{dy}{dt} = 0$$

Meaning at $(0,1)$ we have $x' = 3$ and $y' = 0$ which means the solution is moving to the right. Everything else is filled in according to rules of compatibility.



If we were to fill in more solutions it starts to look pretty:



Example: Consider the system

$$\frac{dx}{dt} = -x + y + x^2$$
$$\frac{dy}{dt} = y - 2xy$$

To find the stationary points we set $-x + y + x^2 = 0$ and $y - 2xy = y(1 - 2x) = 0$. The latter gives us $y = 0$ or $x = \frac{1}{2}$. If $y = 0$ then the former gives us $x = 0, 1$ and if $x = \frac{1}{2}$ then the former gives us $y = \frac{1}{4}$. So there are three stationary points at $(0, 0)$, $(1, 0)$ and $\left(\frac{1}{2}, \frac{1}{4}\right)$.
Noting that $H_y = -x + y + x^2$ and $-H_x = y - 2xy$, so $H_x = 2xy - y$, we get the Hessian:

$$\partial^2 H = \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix} = \begin{bmatrix} 2y & 2x - 1 \\ 2x - 1 & 1 \end{bmatrix}$$

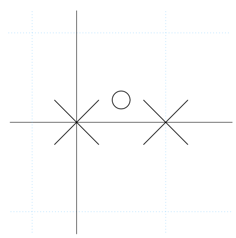
Then at each point:

$$\det \partial^2 H(0, 0) = \det \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = -1 \text{ so } (0, 0) \text{ is a saddle.}$$

$$\det \partial^2 H(1, 0) = \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = -1 \text{ so } (1, 0) \text{ is a saddle.}$$

$$\det \partial^2 H\left(\tfrac{1}{2}, \tfrac{1}{4}\right) = \det \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \tfrac{1}{2} \text{ so } \left(\tfrac{1}{2}, \tfrac{1}{4}\right) \text{ is a center.}$$

A preliminary picture is:



We need to know what direction everything goes in. Interestingly, for this picture, we can find everything out by testing one point in the system. At the point $\left(\frac{1}{2}, 0\right)$ we have:

$$\frac{dx}{dt} = -$$
$$\frac{dy}{dt} = 0$$

Meaning at $\left(\frac{1}{2}, 0\right)$ the solution is moving to the left. Everything else is filled in according to rules of compatibility. Some adjustment of the saddles is also needed!