MATH310 Exam 1 Sample Questions Solutions

1. Which of the following are sets?

   (a) $\{\{\{\}\}\}$
   Solution:
   Yes.

   (b) $\{\mathbb{Z}, \emptyset\}$
   Solution:
   Yes.

   (c) $\mathbb{Z}, \mathbb{R}$
   Solution:
   No.

   (d) $\{1, 2, 3, ...\}$
   Solution:
   Yes.

2. For each of the following sets A and B determine if A is a subset of B. If not, find an element in A which is not in B.

   (a) $A = \{5, 6\}$ and $B = \{4, 5, 6\}$
   Solution:
   Yes.

   (b) $A = \{3x \,\big|\, x \in \mathbb{Z}\}$ and $B = \{6x \,\big|\, x \in \mathbb{Z}\}$
   Solution:
   No, for example $3 \in A$ but $3 \notin B$.

3. Give examples of sets A, B, C such that $A \subseteq B$, $B \subseteq C$ and $A \in C$.

   Solution:

   For example $A = \{1\}$, $B = \{1, 2\}$, $C = \{1, 2, \{1\}\}$.

4. Let $S = \{1, 2, 3, 4, 5\}$. Describe the set $\{3, 5, 7, 9\}$ in the form $\{f(x) \,\big|\, x \in S \text{ and } p(x)\}$ for some function $f(x)$ and open sentence $p(x)$.

   Solution:

   We can write $\{3, 5, 7, 9\} = \{2x + 1 \,\big|\, x \in S \text{ and } x \leq 4\}$.

5. Use a truth table to show that $P \to (Q \wedge R) \not\equiv (P \to Q) \wedge R$.
   Note: You don't have to draw every row of the truth table, just enough rows get the job done.

   Solution:

   Here are all the rows:

P	Q	R	$Q \wedge R$	$P \to Q$	$P \to (Q \wedge R)$	$(P \to Q) \wedge R$
T	T	T	T	T	T	T
T	T	F	F	T	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	T	T	T	T
F	T	F	F	T	T	F
F	F	T	F	T	T	T
F	F	F	T	T	T	F

6. Consider the open sentences over the domain $\mathbb{R}$

$$P(x) : x - 1 \geq 0 \text{ and } Q(x) : x^2 - 9 \geq 0$$

   Find all x such that $P(x) \vee Q(x)$ is true. List as intervals.

   Solution:

   $Q(x)$ is true when $(x-3)(x+3) \geq 0$. This will happen either when both $x - 3 \geq 0$ and $x + 3 \geq 0$, in other words $x \geq 3$ and $x \geq -3$, meaning $x \geq 3$, or when both $x - 3 \leq 0$ and $x + 3 \leq 0$, in other words $x \leq 3$ and $x \leq -3$, meaning $x \leq -3$.

   Thus $Q(x)$ is true when $x \geq 3$ or $x \leq -3$.

   For $P(x) \vee Q(x)$ to be true we have three cases:

   - $P(x)$ and $Q(x)$ both true, meaning $x \geq 1$ and either $x \geq 3$ or $x \leq -3$. This can only happen when $x \geq 3$. This yields $[3, \infty)$.
   - $P(x)$ true but $Q(x)$ false, meaning $x \geq 1$. This yields $[1, \infty)$.
   - $P(x)$ false but $Q(x)$ true, meaning $x \geq 3$ or $x \leq -3$. This yields $(-\infty, -3] \cup [3, \infty)$.

7. Determine if the following are true or false, with justification.

   (a) $\exists x \in \{1, 2, 3\}, 5x - 1$ is divisible by 3.

   Solution:

   True, for example when $x = 2$ then $5(2) - 1 = 9$ is divisible by 3.

   (b) $\forall x \in \mathbb{N}, 2x + 1$ is prime.

   Solution:

   False, for example when $x = 4$ we have $2(4) + 1 = 9$ not prime.

   (c) $\forall x \in \mathbb{N}, \exists y \in \mathbb{R}, y^2 = x$.

   Solution:

   True, since $y = \sqrt{x}$ will work.

8. Prove $\forall x \in \mathbb{R}$, $|2-x| - x \geq -2$.

   Solution:

   We'll look at two cases:

   - If $2 - x \geq 0$ then $|2-x| = 2-x$ and $x \leq 2$ and so $|2-x| - x = 2 - x - x = 2 - 2x \geq 2 - 2(2) = -2$.
   - If $2 - x < 0$ then $|2-x| = -(2-x) = x-2$ and $x > 2$ and so $|2-x| - x = x - 2 - x = -2 \geq -2$.

9. Prove $\forall x, y \in \mathbb{Z}$, $x+y$ is odd if and only if x and y have opposite parity.

   Solution:

   For the forward direction, assume $x+y$ is odd and we claim x and y have opposite parity. To prove this we'll use the contrapositive and assume x and y have the same parity. If x and y are both odd then $x = 2k+1$ and $y = 2j+1$ for $k, j \in \mathbb{Z}$ and then $x+y = 2k+1+2j+1 = 2(k+j+1)$ which is even. If x and y are both even then $x = 2k$ and $y = 2j$ for $k, j \in \mathbb{Z}$ and then $x+y = 2k+2j = 2(k+j)$ which is even.

   For the backward direction, assume x and y have opposite parity and we claim $x+y$ is odd. If x is odd and y is even then $x = 2k+1$ and $y = 2j$ for $k, j \in \mathbb{Z}$ and then $x+y = 2k+1+2j = 2(k+j)+1$ which is odd. If x is even and y is odd then $x = 2k$ and $y = 2j+1$ for $k, j \in \mathbb{Z}$ and then $x+y = 2k+2j+1 = 2(k+j)+1$ which is odd.

10. Are the following statements true or false?

    (a) $\{1\} \in \{1, 2, 3\}$

    Solution:

    False.

    (b) $\emptyset \in \{\}$

    Solution:

    False.

    (c) $\emptyset \in \mathcal{P}(\{1, 2, 3\})$

    Solution:

    True.

11. Write the elements in $\mathcal{P}(\mathcal{P}(\{1\}))$.

    Solution:

    We have:

    $$\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$$

    Then we have:

    $$\mathcal{P}(\mathcal{P}(\{1\}) = \mathcal{P}(\{\emptyset, \{1\}\}) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}$$

12. Let $S = \{0, 3, 6, 9, 12\}$. Describe the set $\{1, 2, 3, 4\}$ in the form $\{f(x) \,\big|\, x \in S \text{ and } p(x)\}$ for some function $f(x)$ and open sentence $p(x)$.

    Solution:

    We can write $\{1, 2, 3, 4\} = \{x/3 \,\big|\, x \in S \text{ and } x \geq 3\}$.

13. Fill in the following truth table only for the possibilities given.

P	Q	R	$P \wedge Q$	$(P \wedge Q) \to R$	$R \to (P \wedge Q)$
T	T	T	T	T	T
T	F	T	F	T	F
F	T	F	F	T	T
F	F	F	F	T	T

    Solution:

14. Consider the open sentences over the domain $\mathbb{R}$

$$P(x) : x - 1 \geq 0 \text{ and } Q(x) : x^2 + 3x \leq 0$$

    Find all $x \in \mathbb{R}$ such that $P(x) \to Q(x)$ is true. List as intervals.

    Solution:

15. Prove $\forall x \in \mathbb{Z}$, a is even iff a^2 is even.

    Solution:

    For the forward direction if a is even then $a = 2k$ for some $k \in \mathbb{Z}$ and then $a^2 = (2k)^2 = 4k^2 = 2(2k^2)$ which is even.

    For the backward direction we use the contrapositive. We assume a is odd and show a^2 is odd. If a is odd then $a = 2k+1$ for some $k \in \mathbb{Z}$ and then $a^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$ which is odd.

16. Prove that if $A \subseteq B$, $B \subseteq C$ and $C \subseteq A$ then $A = B$ and $B = C$.

    Solution:

    First we show $A = B$. If $x \in A$ then since $A \subseteq B$ then $x \in B$. If $x \in B$ then since $B \subseteq C$ we have $x \in C$ and since $C \subseteq A$ we have $x \in A$.

    Next we show $B = C$. The proof is similar.

17. Prove that:

$$\left\{x \in \mathbb{R} \,\middle|\, |x| = 6 - |2x|\right\} = \{-2, 2\}$$

Solution:

First we show $\left\{x \in \mathbb{R} \,\middle|\, |x| = 6 - |2x|\right\} \subseteq \{-2, 2\}$. Suppose $x \in \left\{x \in \mathbb{R} \,\middle|\, |x| = 6 - |2x|\right\}$. We look at two cases.

- If $x \geq 0$ then $|x| = x$ and $|2x| = 2x$ and then since $x = 6 - 2x$ we get $3x = 6$ and so $x = 2$.
- If $x < 0$ then $|x| = -x$ and $|2x| = -2x$ and then since $-x = 6 - (-2x)$ we get $-3x = 6$ and so $x = -2$.

Next we show $\{-2, 2\} \subseteq \left\{x \in \mathbb{R} \,\middle|\, |x| = 6 - |2x|\right\}$. Suppose $x \in \{-2, 2\}$. If $x = -2$ then $|x| = 2$ and $6 - |2x| = 6 - |2(-2)| = 6 - 4 = 2$ and if $x = 2$ then $|x| = 2$ and $6 - |2(2)| = 6 - 4 = 2$.