MATH310 Exam 1 Sample Questions Solutions

1. Which of the following are sets?
   (a) \{\{\}\}\n       Solution:  
       Yes.
   (b) \{\mathbb{Z}, \emptyset\}\n       Solution:  
       Yes.
   (c) \mathbb{Z}, \mathbb{R}\n       Solution:  
       No.
   (d) \{1, 2, 3, \ldots\}\n       Solution:  
       Yes.

2. For each of the following sets \(A\) and \(B\) determine if \(A\) is a subset of \(B\). If not, find an element in \(A\) which is not in \(B\).
   (a) \(A = \{5, 6\}\) and \(B = \{4, 5, 6\}\)
       Solution:  
       Yes.
   (b) \(A = \{3|\, x \in \mathbb{Z}\}\) and \(B = \{6|\, x \in \mathbb{Z}\}\)
       Solution:  
       No, for example 3 \(\in A\) but 3 \(\notin B\).

3. Give examples of sets \(A, B, C\) such that \(A \subseteq B, B \subseteq C\) and \(A \in C\).
   Solution:  
   For example \(A = \{1\}, B = \{1, 2\}, C = \{1, 2, \{1\}\}\).

4. Let \(S = \{1, 2, 3, 4, 5\}\). Describe the set \(\{3, 5, 7, 9\}\) in the form \(\{f(x) \mid x \in S \text{ and } p(x)\}\) for some function \(f(x)\) and open sentence \(p(x)\).
   Solution:  
   We can write \(\{3, 5, 7, 9\} = \{2x + 1 \mid x \in S \text{ and } x \leq 4\}\).
5. Use a truth table to show that \( P \rightarrow (Q \land R) \not\equiv (P \rightarrow Q) \land R \).
Note: You don’t have to draw every row of the truth table, just enough rows get the job done.

**Solution:**
Here are all the rows:

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6. Consider the open sentences over the domain \( \mathbb{R} \):

\( P(x) : x - 1 \geq 0 \) and \( Q(x) : x^2 - 9 \geq 0 \)

Find all \( x \) such that \( P(x) \lor Q(x) \) is true. List as intervals.

**Solution:**
\( Q(x) \) is true when \( (x - 3)(x + 3) \geq 0 \). This will happen either when both \( x - 3 \geq 0 \) and \( x + 3 \geq 0 \), in other words \( x \geq 3 \) and \( x \geq -3 \), meaning \( x \geq 3 \), or when both \( x - 3 \leq 0 \) and \( x + 3 \leq 0 \), in other words \( x \leq 3 \) and \( x \leq -3 \), meaning \( x \leq -3 \).

Thus \( Q(x) \) is true when \( x \geq 3 \) or \( x \leq -3 \).

For \( P(x) \lor Q(x) \) to be true we have three cases:

- \( P(x) \) and \( Q(x) \) both true, meaning \( x \geq 1 \) and either \( x \geq 3 \) or \( x \leq -3 \). This can only happen when \( x \geq 3 \). This yields \([3, \infty)\).
- \( P(x) \) true but \( Q(x) \) false, meaning \( x \geq 1 \). This yields \([1, \infty)\).
- \( P(x) \) false but \( Q(x) \) true, meaning \( x \geq 3 \) or \( x \leq -3 \). This yields \((-\infty, -3] \cup [3, \infty)\).

7. Determine if the following are true or false, with justification.

(a) \( \exists x \in \{1, 2, 3\}, 5x - 1 \) is divisible by 3.

**Solution:**
True, for example when \( x = 2 \) then \( 5(2) - 1 = 9 \) is divisible by 3.

(b) \( \forall x \in \mathbb{N}, 2x + 1 \) is prime.

**Solution:**
False, for example when \( x = 4 \) we have \( 2(4) + 1 = 9 \) not prime.

(c) \( \forall x \in \mathbb{N}, \exists y \in \mathbb{R}, y^2 = x \).

**Solution:**
True, since \( y = \sqrt{x} \) will work.
8. Prove $\forall x \in \mathbb{R}, |2 - x| - x \geq -2$.

**Solution:**
We'll look at two cases:

- If $2 - x \geq 0$ then $|2 - x| = 2 - x$ and $x \leq 2$ and so $|2 - x| - x = 2 - x - x = 2 - 2x \geq 2 - 2(2) = -2$.
- If $2 - x < 0$ then $|2 - x| = -(2 - x) = x - 2$ and $x > 2$ and so $|2 - x| - x = x - 2 - x = -2 \geq -2$.

9. Prove $\forall x, y \in \mathbb{Z}, x + y$ is odd if and only if $x$ and $y$ have opposite parity.

**Solution:**
For the forward direction, assume $x + y$ is odd and we claim $x$ and $y$ have opposite parity. To prove this we'll use the contrapositive and assume $x$ and $y$ have the same parity. If $x$ and $y$ are both odd then $x = 2k+1$ and $y = 2j+1$ for $k, j \in \mathbb{Z}$ and then $x+y = 2k+1+2j+1 = 2(k+j+1)$ which is even. If $x$ and $y$ are both even then $x = 2k$ and $y = 2j$ for $k, j \in \mathbb{Z}$ and then $x+y = 2k+2j = 2(k+j)$ which is even.

For the backward direction, assume $x$ and $y$ have opposite parity and we claim $x+y$ is odd. If $x$ is odd and $y$ is even then $x = 2k+1$ and $y = 2j$ for $k, j \in \mathbb{Z}$ and then $x+y = 2k+1+2j = 2(k+j)+1$ which is odd. If $x$ is even and $y$ is odd then $x = 2k$ and $y = 2j+1$ for $k, j \in \mathbb{Z}$ and then $x+y = 2k+2j+1 = 2(k+j)+1$ which is odd.

10. Are the following statements true or false?

(a) $\{1\} \in \{1, 2, 3\}$

**Solution:**
False.

(b) $\emptyset \in \{}$

**Solution:**
False.

(c) $\emptyset \in P(\{1, 2, 3\})$

**Solution:**
True.

11. Write the elements in $P(P(\{1\}))$.

**Solution:**
We have:

$$P(\{1\}) = \{\emptyset, \{1\}\}$$

Then we have:

$$P(P(\{1\}) = P(\{\emptyset, \{1\}\}) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \emptyset, \{1\}\})$$
12. Let $S = \{0, 3, 6, 9, 12\}$. Describe the set $\{1, 2, 3, 4\}$ in the form $\{f(x) \mid x \in S \text{ and } p(x)\}$ for some function $f(x)$ and open sentence $p(x)$.

**Solution:**
We can write $\{1, 2, 3, 4\} = \{x/3 \mid x \in S \text{ and } x \geq 3\}$.

13. Fill in the following truth table only for the possibilities given.

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**Solution:**

14. Consider the open sentences over the domain $\mathbb{R}$

$$P(x) : x - 1 \geq 0 \text{ and } Q(x) : x^2 + 3x \leq 0$$

Find all $x \in \mathbb{R}$ such that $P(x) \rightarrow Q(x)$ is true. List as intervals.

**Solution:**

15. Prove $\forall x \in \mathbb{Z}$, $a$ is even iff $a^2$ is even.

**Solution:**
For the forward direction if $a$ is even then $a = 2k$ for some $k \in \mathbb{Z}$ and then $a^2 = (2k)^2 = 4k^2 = 2(2k^2)$ which is even.

For the backward direction we use the contrapositive. We assume $a$ is odd and show $a^2$ is odd. If $a$ is odd then $a = 2k+1$ for some $k \in \mathbb{Z}$ and then $a^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$ which is odd.

16. Prove that if $A \subseteq B$, $B \subseteq C$ and $C \subseteq A$ then $A = B$ and $B = C$.

**Solution:**
First we show $A = B$. If $x \in A$ then since $A \subseteq B$ then $x \in B$. If $x \in B$ then since $B \subseteq C$ we have $x \in C$ and since $C \subseteq A$ we have $x \in A$.

Next we show $B = C$. The proof is similar.
17. Prove that:
\[ \{ x \in \mathbb{R} \mid |x| = 6 - |2x| \} = \{-2, 2\} \]

Solution:
First we show \( \{ x \in \mathbb{R} \mid |x| = 6 - |2x| \} \subseteq \{-2, 2\} \). Suppose \( x \in \{ x \in \mathbb{R} \mid |x| = 6 - |2x| \} \). We look at two cases.

- If \( x \geq 0 \) then \( |x| = x \) and \( |2x| = 2x \) and then since \( x = 6 - 2x \) we get \( 3x = 6 \) and so \( x = 2 \).
- If \( x < 0 \) then \( |x| = -x \) and \( |2x| = -2x \) and then since \( -x = 6 - (-2x) \) we get \( -3x = 6 \) and so \( x = -2 \).

Next we show \( \{-2, 2\} \subseteq \{ x \in \mathbb{R} \mid |x| = 6 - |2x| \} \). Suppose \( x \in \{-2, 2\} \). If \( x = -2 \) then \( |x| = 2 \) and \( 6 - |2x| = 6 - |2(-2)| = 6 - 4 = 2 \) and if \( x = 2 \) then \( |x| = 2 \) and \( 6 - |2(2)| = 6 - 4 = 2 \).