MATH310 Exam 2 Sample Questions

1. Let A and B be sets. Prove that $A \cup B \subseteq B$ iff $A \subseteq B$.
2. Let A and B be sets. Prove that $A = (A - B) \cup (A \cap B)$.
3. The following are false. Provide counterexamples as evidence:
   (a) $\forall x \in \mathbb{R}$ with $x > 0$, $x^2 \geq x$.
   (b) $\forall a, b, c, n \in \mathbb{N}$ with $n > 1$ and n not dividing c if n divides $ac - bc$ then n divides $a - b$.
   (c) $\forall a, b, c \in \mathbb{Z}$, if $a \mid b$ and $b \nmid c$ then $a \nmid c$.
   (d) $\forall$ sets A, B, if $A \cap B = \emptyset$ then $A = \emptyset$ or $B = \emptyset$.
4. Prove that if $a, b \in \mathbb{Z}$ with $a \geq 2$ then $a \nmid b$ or $a \nmid (b + 1)$.
5. Prove that the sum of the squares of two odd integers cannot be the square of an integer.
6. Indicate what you would assume when proving each of the following by contradiction. You do not need to prove either!
   (a) $\forall x, (P(x) \vee Q(x)) \rightarrow (R(x) \wedge (\sim S(x)))$
   (b) $P \rightarrow (Q \rightarrow R)$
7. Define $f(x) = x^5 + x^2 + 1$.
   (a) Prove that there is some $x \in \mathbb{R}$ with $f(x) = 0$.
   (b) Prove that there is no $x \in \mathbb{R}$ with $x \geq 0$ and $f(x) = 0$.
8. Prove using weak induction that $\forall n \in \mathbb{N}$ we have
$$1(1!) + 2(2!) + ... + n(n!) = (n + 1)! - 1$$
9. Prove using weak induction that $\forall n \in \mathbb{N}$ with $n \geq 4$ that $3^n > 5n^2$.
10. Define a sequence recursively by $a_1 = 1$, $a_2 = 3$ and $a_n = 2a_{n-1} + a_{n-2}$. Prove using strong induction that a_n is odd for all integers $n \geq 1$.
11. Prove using strong induction that any postage 18 cents or greater can be made using only 4 cent and 7 cent stamps.
12. Prove that this statement is true: $\exists n \in \mathbb{Z}$, $n^3 < n$.
13. Prove that this statement is false: $\forall$ sets A, B if $A \subseteq B$ then $A \cap B \neq B$
14. Prove or disprove the statement:
$$\forall n \in \mathbb{N}, 4^n > n^4$$
15. Define the relation R on $\mathbb{Z}$ by aRb if $3 \nmid (a + 2b)$. Prove that R is not an equivalence relation.
16. Define the relation R on $\mathbb{Z}$ by aRb if $|a - b| \leq 10$. Prove that R is not an equivalence relation.
17. Prove that $f : (\mathbb{R} - \{1\}) \rightarrow (\mathbb{R} - \{2\})$ given by $f(x) = \frac{2x+1}{x-1}$ is invertible and find $f^{-1}(y)$.

18. Suppose $A, B \subseteq U$. Prove that: $\chi_{A\cup B}(x) = 1$ iff $\chi_A(x) + \chi_B(x) > 0$

19. Suppose f and g are two functions with the same domain D. Define $A = \{x \in D \,\big|\, f(x) = g(x)$.
    Prove $A = D$ iff $f = g$.

20. Prove that the function $f(x) = x^2 - x$ for $x \geq 1$ is increasing.

21. Prove that the function $f : (0, \infty) \to (1, \infty)$ given by $f(x) = \frac{x+1}{x}$ is surjective.

22. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 5$ is not surjective.

23. Prove that the function $f : (0, \infty) \to (1, \infty)$ given by $f(x) = \frac{x+1}{x}$ is injective.

24. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + |x|$ is not injective.

25. Prove that the function $f : (\mathbb{R} - \{0\}) \to (\mathbb{R} - \{1\})$ defined by $f(x) = \frac{x-1}{x}$. is 1-1 and find a formula for its inverse. [20 pts]

26. Suppose A is a set with a elements and B is a set with b elements. Prove that if $f : A \to B$ is bijective then $a = b$.

27. Prove that if $f : A \to B$ and $g : B \to C$ are both surjective then so is $g \circ f : A \to C$.