MATH310 Exam 2 Sample Questions Solutions

1. Let A and B be sets. Prove that $A \cup B \subseteq B$ iff $A \subseteq B$.

   Solution:

   $\longrightarrow$:

   Assume $A \cup B \subseteq B$. We claim $A \subseteq B$. Let $x \in A$. Then $A \in A \cup B$ and since $A \cup B \subseteq B$ we know $x \in B$.

   $\longleftarrow$:

   Assume $A \subseteq B$. We claim $A \cup B \subseteq B$. Let $x \in A \cup B$. Then either $x \in A$ or $x \in B$ or both. If $x \in A$ then since $A \subseteq B$ we know $x \in B$. If $x \in B$ then $x \in B$. Either way, $x \in B$.

2. Let A and B be sets. Prove that $A = (A - B) \cup (A \cap B)$.

   Solution:

   $\subseteq$:

   Suppose $x \in A$. We claim $x \in (A-B) \cup (A \cap B)$. There are two cases to look at. If $x \notin B$ then $x \in A - B$ and so $x \in (A-B) \cup (A \cap B)$. If $x \in B$ then $x \in A \cap B$ and so $x \in (A-B) \cup (A \cap B)$.

   $\supseteq$:

   Suppose $x \in (A-B) \cup (A \cap B)$. We claim $x \in A$. There are two cases to look at. If $x \in A - B$ then $x \in A$. If $x \in A \cap B$ then $x \in A$. Either way, $x \in A$.

3. The following are false. Provide counterexamples as evidence:

   (a) $\forall x \in \mathbb{R}$ with $x > 0$, $x^2 \geq x$.
   Solution:
   A counterexample is $x = 0.1$.

   (b) $\forall a, b, c, n \in \mathbb{N}$ with $n > 1$ and n not dividing c if n divides $ac - bc$ then n divides $a - b$.
   Solution:
   A counterexample is $n = 10$, $a = 4$, $b = 2$ and $c = 5$.

   (c) $\forall a, b, c \in \mathbb{Z}$, if $a \mid b$ and $b \nmid c$ then $a \nmid c$.
   Solution:
   A counterexample is $a = 2$, $b = 4$ and $c = 2$.

   (d) $\forall$ sets A, B, if $A \cap B = \emptyset$ then $A = \emptyset$ or $B = \emptyset$.
   Solution:
   A counterexample is $A = \{1\}$ and $B = \{2\}$.

4. Prove that if $a, b \in \mathbb{Z}$ with $a \geq 2$ then $a \nmid b$ or $a \nmid (b+1)$.

   Solution:

   Note: We are trying to prove $P \vee Q$. We will do it by contradiction and assume $\sim P \wedge \sim Q$.

   Assume $a \mid b$ and $a \mid (b+1)$. Since $a \mid b$ we know $ak = b$ for some $k \in \mathbb{Z}$ and since $a \mid (b+1)$ we know $aj = b + 1$ for some $j \in \mathbb{Z}$.

   Then $aj = b + 1 = ak + 1$ so then $a(j - k) = 1$ so then $a \mid 1$ but $a \geq 2$ which makes this impossible.

5. Prove that the sum of the squares of two odd integers cannot be the square of an integer.

   Solution:

   Assume our two odd integers are $2k+1$ and $2j+1$ for $k, j \in \mathbb{Z}$. The sum of the squares is $(2k+1)^2 + (2j+1)^2 = 4k^2 + 4k + 1 + 4j^2 + 4j + 1 = 4(k^2 + k + j^2 + j) + 2$.

   This cannot be the square of an integer because the square of an odd integer has the form $(2a+1)^2 = 4a^2 + 4a + 1 = 4(a^2 + a) + 1$ and the square of an even integer has the form $(2a)^2 = 4(a^2)$.

6. Indicate what you would assume when proving each of the following by contradiction. You do not need to prove either!

   (a) $\forall x, (P(x) \vee Q(x)) \rightarrow (R(x) \wedge (\sim S(x)))$
   Solution:
   We would assume:

   $$\sim [\forall x, (P(x) \vee Q(x)) \rightarrow (R(x) \wedge (\sim S(x)))]$$

   This is equivalent to:

   $$\exists x, (P(x) \vee Q(x)) \wedge (\sim R(x) \vee S(x))$$

   (b) $P \rightarrow (Q \rightarrow R)$
   Solution:
   We would assume:

   $$\sim [P \rightarrow (Q \rightarrow R)]$$

   This is equivalent to:

   $$P \wedge (Q \wedge \sim R)$$

7. Define $f(x) = x^5 + x^2 + 1$.

   (a) Prove that there is some $x \in \mathbb{R}$ with $f(x) = 0$.
   Solution:
   Observe that $f(-2) = -32 + 4 + 1 < 0$ and $f(0) = 1$ and since $f(x)$ is continuous the intermediate value theorem tells us there is some $x \in (-2, 0)$ with $f(x) = 0$.

   (b) Prove that there is no $x \in \mathbb{R}$ with $x \geq 0$ and $f(x) = 0$.
   Solution:
   You can ignore this since we've stayed away from too much calculus, but essentially it's because $f(0) = 1$ and $f'(x) = 4x^2 + 2x$ which is positive, so $f(x)$ is increasing, when $x > 0$.

8. Prove using weak induction that $\forall n \in \mathbb{N}$ we have

   $$1(1!) + 2(2!) + ... + n(n!) = (n+1)! - 1$$

   Solution:

For the base case observe that the left side is $1(1!) = 1$ and the right side is $(1+1)! - 1 = 1$, so they are equal.

For the inductive step we assume that $1(1!) + 2(2!) + ... + n(n!) = (n+1)! - 1$ and we claim that $1(1!) + 2(2!) + ... + n(n!) + (n+1)(n+1)! = ((n+1)+1)! - 1$.

To this end, observe that:

$$\begin{aligned} 1(1!) + 2(2!) + ... + n(n!) + (n+1)(n+1)! &= (n+1)! - 1 + (n+1)(n+1)! \\ &= (n+1)!(1+n+1) - 1 \\ &= (n+2)(n+1)! - 1 \\ &= (n+2)! - 1 \\ &= ((n+1)+1)! - 1 \end{aligned}$$

9. Prove using weak induction that $\forall n \in \mathbb{N}$ with $n \geq 4$ that $3^n > 5n^2$.

   Solution:

   For the base case observe that the left side is $3^4 = 81$ and the right side is $5(4)^2 = 80$, so it is true.

   For the inductive step we assume that $3^n > 5n^2$ and we claim that $3^{n+1} > 5(n+1)^2$.

   We will actually show that $3^{n+1} - 5(n+1)^2 > 0$.

   Observe that:

$$\begin{aligned} 3^{n+1} - 5(n+1)^2 &= 3 \cdot 3^n - 5n^2 - 10n - 5 \\ &\geq 3(5n^2) - 5n^2 - 10n - 5 \\ &\geq 10n^2 - 10n - 5 \\ &\geq 10n(n-1) - 5 \end{aligned}$$

   Since $n \geq 4$ we have $10n(n-1) - 5 \geq 40(3) - 5$ which is certainly greater than 0.

10. Define a sequence recursively by $a_1 = 1$, $a_2 = 3$ and $a_n = 2a_{n-1} + a_{n-2}$. Prove using strong induction that a_n is odd for all integers $n \geq 1$.

    Solution:

    For the inductive step assume that all of $a_1, a_2, ..., a_n$ are oddd. We claim a_{n+1} is odd.

    Observe that for some $k, j \in \mathbb{Z}$ we have:

$$\begin{aligned} a_{n+1} &= 2a_n + a_{n-1} \\ &= 2(2k+1) + 2j + 1 \\ &= 2(2k+j+1) + 1 \end{aligned}$$

    This is odd.

    For the base case(s) note that we reference $n-1$ so we must have $n-1 \geq 1$ and so $n \geq 2$. Thus the $n = 1, 2$ cases must be done separately. But $a_1 = 1$ and $a_2 = 3$ are both odd, so we are done.

11. Prove using strong induction that any postage 18 cents or greater can be made using only 4 cent and 7 cent stamps.

    Solution:

    For the inductive step assume we can do $18, 19, ..., n$ cents. Since we can do $n-3$ we just do that, then add a 4-cent stamp.

    For the base case(s) note that we reference $n-3$ so we must have $n-3 \geq 18$ and so $n \geq 21$. Thus the $n = 18, 19, 20, 21$ cases must be done separately.

    Observe that $18 = 2(7) + 1(4)$, $19 = 1(7) + 3(4)$, $20 = 0(7) + 5(4)$, and $21 = 3(7) + 0(4)$.

12. Prove that this statement is true: $\exists n \in \mathbb{Z}$, $n^3 < n$.

    Solution:

    Obseve that $n = -2$ works.

13. Prove that this statement is false: $\forall$ sets A, B if $A \subseteq B$ then $A \cap B \neq B$

    Solution:

    Observe that a counterexample is $A = \{1\}$ and $B = \{1, 2\}$.

14. Prove or disprove the statement:

    $$\forall n \in \mathbb{N}, 4^n > n^4$$

    Solution:

    It is false since a counterexample is $n = 4$.

15. Define the relation R on $\mathbb{Z}$ by aRb if $3 \nmid (a + 2b)$. Prove that R is not an equivalence relation.

    Solution:

    Observe that $0R1$ is true since $3 \nmid (0 + 2(1)$ and $1R3$ is true since $3 \nmid (1 + 2(3))$ but $0R3$ is false since $3 \mid (0 + 2(3))$.

16. Define the relation R on $\mathbb{Z}$ by aRb if $|a - b| \leq 10$. Prove that R is not an equivalence relation.

    Solution:

    Observe that $10R0$ and $20R10$ are true but $20R0$ is not, so transitivity fails.

17. Prove that $f : (\mathbb{R} - \{1\}) \to (\mathbb{R} - \{2\})$ given by $f(x) = \frac{2x+1}{x-1}$ is invertible and find $f^{-1}(y)$.

    Solution:

    Observe that f is 1-1 since if we have $x_1, x_2 \in \mathbb{R} - \{1\}$ with $f(x_1) = f(x_2)$ then:

    $$\begin{aligned}
    \frac{2x_1 + 1}{x_1 - 1} &= \frac{2x_2 + 1}{x_2 - 1} \\
    (2x_1 + 1)(x_2 - 1) &= (2x_2 + 1)(x_1 - 1) \\
    2x_1x_2 - 2x_1 + x_2 - 1 &= 2x_1x_1 - 2x_2 + x_1 - 1 \\
    3x_2 &= 3x_1 \\
    x_2 &= x_1
    \end{aligned}$$

18. Suppose $A, B \subseteq U$. Prove that: $\chi_{A\cup B}(x) = 1$ iff $\chi_A(x) + \chi_B(x) > 0$

    Solution:

    $\longrightarrow$:

    Suppose $\chi_{A\cup B}(x) = 1$ so then $x \in A \cup B$ so then $x \in A$ or $x \in B$ or both. It follows that $\chi_A(x) = 1$ or $\chi_B(x) = 1$ or both. Together $\chi_A(x) + \chi_B(x) > 0$.

    $\longleftarrow$:

    By contrapositive, suppose $\chi_A(x) + \chi_B(x) = 0$ so then $\chi_A(x) = 0$ and $\chi_B(x) = 0$ so then $x \notin A$ and $x \notin B$. It follows that $A \notin A \cup B$ and so $\chi_{A\cup b}(x) = 0$.

19. Suppose f and g are two functions with the same domain D. Define $A = \{x \in D \,\Big|\, f(x) = g(x)$. Prove $A = D$ iff $f = g$.

    Solution:

    $\longrightarrow$:

    Suppose $A = D$. This means that $f(x) = g(x)$ for all $x \in D$ and so $f = g$.

    $\longleftarrow$:

    Suppose $f = g$. Let $x \in D$, so then $f(x) = g(x)$ and so $x \in A$. Thus $D \subseteq A$. Since $A \subseteq D$ by definition, we have $A = D$.

20. Prove that the function $f(x) = x^2 - x$ for $x \geq 1$ is increasing.

    Solution:

    Suppose $x_1, x_2 \in (1, \infty)$. We'll show if $x_1 < x_2$ then $x_1^2 - x_1 < x_2^2 - x_2$.

    Since $x_1 < x_2$ we have $x_1 - 1 < x_2 - 1$. Since all of $x_1, x_2, x_1 - 1, x_2 - 1$ are positive we then have:

    $$x_1(x_1 - 1) < x_2(x_2 - 1)$$

    This is exactly our claim.

21. Prove that the function $f : (0, \infty) \to (1, \infty)$ given by $f(x) = \frac{x+1}{x}$ is surjective.

    Solution:

    For any $y \in (1, \infty)$ we need some x with $f(x) = y$. this means finding an x with:

    $$\begin{aligned} \frac{x+1}{x} &= y \\ x + 1 &= xy \\ x - xy &= -1 \\ xy - x &= 1 \\ x(y-1) &= 1 \\ x &= \frac{1}{y-1} \end{aligned}$$

    Since $y > 1$ we know that $\frac{1}{y-1} \in (0, \infty)$ and we are done.

22. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 5$ is not surjective.

Solution:

There is no x with $f(x) = 0$, since such an x would have $x^2 + 5 = 0$ and so $x^2 = -5$, which is not possible with $x \in \mathbb{R}$.

23. Prove that the function $f : (0, \infty) \to (1, \infty)$ given by $f(x) = \frac{x+1}{x}$ is injective.

Solution:

Suppose $x_1, x_2 \in (0, \infty)$. We'll prove that if $f(x_1) = f(x_2)$ then $x_1 = x_2$:

$$\begin{aligned} \frac{x_1 + 1}{x_1} &= \frac{x_2 + 1}{x_2} \\ (x_1 + 1)(x_2) &= (x_2 + 1)(x_1) \\ x_1 x_2 + x_2 &= x_1 x_2 + x_1 \\ x_2 &= x_1 \end{aligned}$$

24. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + |x|$ is not injective.

Solution:

Observe that $f(0) = 0$ and $f(-1) = 0$.

25. Prove that the function $f : (\mathbb{R} - \{0\}) \to (\mathbb{R} - \{1\})$ defined by $f(x) = \frac{x-1}{x}$. is 1-1 and find a formula for its inverse. [20 pts]

Solution:

Suppose $x_1, x_2 \in \mathbb{R} - \{0\}$. We'll prove that if $f(x_1) = f(x_2)$ then $x_1 = x_2$:

$$\begin{aligned} \frac{x_1 - 1}{x_1} &= \frac{x_2 - 1}{x_2} \\ (x_1 - 1)(x_2) &= (x_2 - 1)(x_1) \\ x_1 x_2 - x_2 &= x_1 x_2 - x_1 \\ -x_2 &= -x_1 \\ x_2 &= x_1 \end{aligned}$$

For the formula we write $y = \frac{x-1}{x}$ and solve for x:

$$\begin{aligned} \frac{x - 1}{x} &= y \\ x - 1 &= xy \\ x - xy &= 1 \\ x(1 - y) &= 1 \\ x &= \frac{1}{1 - y} \end{aligned}$$

Thus the formula is $f^{-1}(y) = \frac{1}{1-y}$.

26. Suppose A is a set with a elements and B is a set with b elements. Prove that if $f : A \to B$ is bijective then $a = b$.

    Solution:

    If $a > b$ then f cannot be $1 - 1$ and if $a < b$ then f cannot be onto.

27. Prove that if $f : A \to B$ and $g : B \to C$ are both surjective then so is $g \circ f : A \to C$.

    Solution:

    Suppose $z \in C$. We need some $x \in A$ with $(x, z) \in g \circ f$.

    Since g is surjective there is some $y \in B$ with $(y, z) \in g$. Since f is surjective there is some $x \in A$ with $(x, y) \in f$.

    By the definition of composition we then have $(x, y) \in g \circ f$.