MATH310 Exam 3 Sample Questions

1. Write down the definition of \( \{ a_n \} \to a_0 \) and then write down its negation with the negation pushed through then simplified.

2. Use the definition to prove:
\[
\left\{ \frac{2n}{1-3n} \right\} \to \frac{2}{3}
\]

3. Use the definition to prove:
\[
\left\{ \frac{5^n - 3}{6^n + 1} \right\} \to 0
\]

4. Use the definition to prove:
\[
\{ (-1)^n \} \not\to 0
\]

5. Use the definition to prove:
\[
\{ \sqrt{n} \} \not\to 5
\]

6. Use the definition to prove that the following sequence does not converge:
\[
\{ \sqrt{n} \}
\]

7. Prove that a sequence of numbers greater than or equal to 1 which converges must converge to a number greater than or equal to 1.

8. Write down the definition of \( \lim_{x \to x_0} f(x) = L \) and then write down its negation with the negation pushed through then simplified.

9. Use the definition to prove:
\[
\lim_{x \to 3} 1 - x = -2
\]

10. Use the definition to prove:
\[
\lim_{x \to -1} \frac{x + 1}{x - 3} = 0
\]

11. Use the definition to prove:
\[
\lim_{x \to 10} 5x + 2 \neq 42
\]

12. Write down the definition of “\( f(x) \) is continuous at \( x = x_0 \)” and then write down its negation with the negation pushed through then simplified.

13. Prove that the following function is not continuous at \( x = 2 \):
\[
f(x) = \begin{cases} 
 4x & \text{if } x \leq 2 \\
 3x^2 & \text{if } x > 2 
\end{cases}
\]

14. Prove that \( f(x) = 3 - 4x \) is continuous at \( x = 10 \).

15. Prove that the following set is countable:
\[
\left\{ (a, b) \left| a, b \in \mathbb{Z} \land a < b \right. \right\}
\]
16. Prove that the set of infinite binary strings like 1010111... is uncountable.

17. Prove that:
   \[ |(-3, \infty)| = |\mathbb{R}| \]

18. Prove that:
   \[ |\mathbb{Z}| = \left| \left\{ (a, b, c) \mid a, b, c \in \mathbb{Z} \right\} \right| \]

19. Prove that if \( A \) is a set with a supremum then the supremum must be unique.

20. Prove that if \( A \) is a set with a maximum then the maximum must be unique.

21. Prove using the definition of open that \((1, 3)\) is open.

22. Prove using the definition of open that \([1, 3)\) is not open.

23. Prove using the definition of open that \((1, \infty)\) is open.

24. Prove using the definition of closed that \((1, 3)\) is not closed.

25. Prove using the definition of closed that \([1, 3]\) is closed.

26. Prove using the definition of closed that \([1, \infty)\) is closed.

27. (Challenging 410 problem) Prove using the definition of closed that \( S = \{0, 5\} \) is closed.

28. Prove using the definition of closed that \( \mathbb{R} \) is closed.

29. Prove using the definition of open that \( \mathbb{R} \) is open.

30. (Challenging 410 problem) Suppose \( f : \mathbb{R} \to \mathbb{R} \) has the property that \( f \) is continuous and \( f(x) = 0 \) for all \( x \in S \), where \( S \subseteq \mathbb{R} \) is dense. Prove that \( f(x) = 0 \) for all \( x \).