Computer Graphics

Justin Wyss-Gallifent

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2.1 Introduction

2.1.1 Chapter Goal

In computer graphics the natural representation of a point is as a vector, meaning the natural way to store the point (x, y) is as $[x; y]$ (in two dimensions) or (x, y, z) is as $[x; y; z]$.

The goal of this section is to try to figure out how to work with points in 2D and 3D in such a way that standard movements (translations and rotations) as well as projections (used to turn 3D pictures into 2D representations) can all be managed using matrix multiplication.

More specifically if $\bar{p}$ is a point in two dimensions then we would like to represent a rotation (for example) as a matrix R such that the vector $R\bar{p}$ is the new point after it has been rotated.

The reason for this is twofold:

- Matrix multiplication is easy to calculate.
- If we have multiple points $\bar{p}_1, ..., \bar{p}_k$ then we can apply M to all of them simultaneously by simply putting $\bar{p}_1, ..., \bar{p}_k$ in a matrix because

$$M\,[\bar{p}_1 \ldots \bar{p}_k] = [M\bar{p}_1 \ldots M\bar{p}_k]$$

- If we have two operations and they are represented by the matrices A and B then the operation "A then B" can be represented easily by the matrix BA in the sense that the product $BA\bar{p} = B(A\bar{p})$ is the point which results from doing A then B to $\bar{p}$.

2.1.2 Brief Review on Linearity

Recall that a function f from $\mathbb{R}^n$ to $\mathbb{R}^m$ is linear if and only if it can be represented by matrix multiplication where the matrix equals $[f(\bar{e}_1) \ldots f(\bar{e}_n)]$.

Some consequences of this are:

- If we have a function and know it's linear then we can construct the matrix as instructed.
- If we have a function and claim it's linear one approach is to find $f(\bar{e}_i)$, construct the matrix and show that the matrix multiplication actually does what f does. The argument here is that f is linear if and only if the matrix would represent it, and since it does, we can conclude that f is linear.
- If we have a function and claim it's not linear one approach is to find $f(\bar{e}_i)$, construct the matrix and show that the matrix multiplication does not do what f does. The argument here is that f is linear if and only if the matrix would represents, and since it doesn't, f isn't linear.

- If we have a function and don't know if it's linear we can't simply construct the matrix and say it is. It's necessary that the matrix multiplication does what f does to finish the job.

2.2 Translations in 2D and Lower

2.2.1 Translation Problem

The first thing we'd like to do is translate a point, meaning shift it horizontally and/or vertically. This means finding a matrix T such that $T\bar{p}$ translates the point.

However this will never work, the reason being that if $\bar{p} = \bar{0}$ then no matter what we choose for T we will get $T\bar{p} = T\bar{0} = \bar{0}$ meaning the origin is never going to be translated no matter what choice we have for T.

The problem is deeper than this, the problem is not just with the origin, the problem is that translations are not linear and matrix multiplications are, so it seems we're out of luck.

So what can we do?

2.2.2 Stepping back to 1D

To see how we can fix this it's helpful to step back to one dimension where the analogy would be that we want to somehow represent the shift $x \to x + 1$ by a matrix multiplication.

Instead of representing x simply as a single variable (a 1×1 vector $[x]$) if we represent it by $\begin{bmatrix} x \\ 1 \end{bmatrix}$ then observe that for any a we have:

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} x + a \\ 1 \end{bmatrix}$$

What is going on here?

The matrix

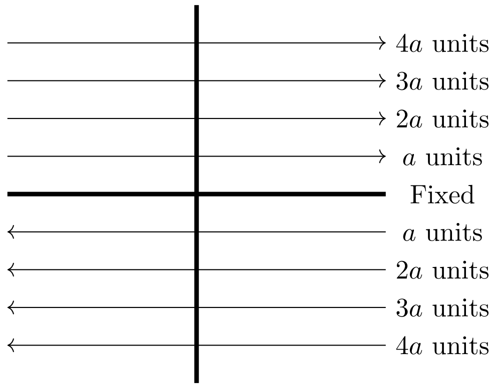
$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

is a *shear transformation.* If we look at what it does to any point:

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y \end{bmatrix}$$

we see that each horizontal line is preserved as whole but moved to the left or right proportional to the particular y value. So for example the line $y = 1$ is moved to the right by a units, the line $y = 2$ is moved to the right by $2a$ units, the line $y = -3$ is moved to the right by $-3a$ units, and so on.

Visually:



What this means then is that a point x in one dimension can be represented by a vector $\begin{bmatrix} x \\ 1 \end{bmatrix}$ and translation by a units can be represented by the matrix

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

Example 2.1. Translation by 3 units is represented by the matrix

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

and then to shift the point $x = 7$ we do:

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

and see that it's shifted to $x = 10$.

2.2.3 Back to 2D and Building a Matrix

So then for our 2D point what we'll do is represent our point by the vector:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

For any a, b we claim that there is a linear transformation f with:

$$f\left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\right) = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

If such a linear transformation exists then it can be represented by a matrix M and that matrix is dictated by what it does to the standard basis. More specifically if such an M exists then it would equal:

$$\left[f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \quad f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \quad f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \right]$$

In order to find these, note that the translation would shift $(1,0)$ to $(1+a,b)$, $(0,1)$ to $(a,1+b)$ and $(0,0)$ to (a,b). In other words:

$$\begin{aligned}
f\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} 1+a \\ b \\ 1 \end{bmatrix} \\
f\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} a \\ 1+b \\ 1 \end{bmatrix} \\
f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}
\end{aligned}$$

It follows then that by linearity we must have:

$$\begin{aligned}
f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) &= f\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \\
&= f\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) - f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} 1+a \\ b \\ 1 \end{bmatrix} - \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

and

$$
\begin{aligned}
f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) &= f\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \\
&= f\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) - f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} a \\ 1+b \\ 1 \end{bmatrix} - \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}
$$

Therefore if such a linear transformation f exists it must be represented by the matrix

$$
M = \left[f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \quad f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \quad f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \right] = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}
$$

So we check that this M does in fact work:

$$
M \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}
$$

Therefore we have a matrix that does the desired job and so in general for any a, b we write:

$$
T(a,b) = \left[M \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad M \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}
$$

Example 2.2. Translation by $+2$ units in the x direction and -5 units in the y direction is represented by the matrix

$$
T(2,-5) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}
$$

So then if we translate the points $(7, 1)$ and $(-3, 0)$ we would represent the points by vectors $[7; 1; 1]$ and $[-3; 0, 1]$ and then:

$$T(2,-5)\begin{bmatrix} 7 & -3 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 7 & -3 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -1 \\ -4 & -5 \\ 1 & 1 \end{bmatrix}$$

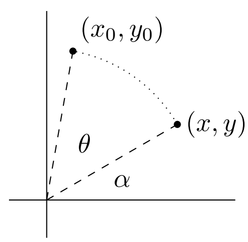
resulting in the points $(9, -4)$ and $(-1, -5)$.

2.3 Rotations in 2D

First we'll deal with rotations around the origin (which is easier) and then we'll see how we can rotate around any point.

Unless otherwise specified all rotations in 2D are counterclockwise.

To rotate around the origin by θ radians consider that we want to take the point (x, y) to (x_0, y_0) as shown:



First note that:

$$\begin{aligned} x &= \sqrt{x^2 + y^2}\cos\alpha \\ y &= \sqrt{x^2 + y^2}\sin\alpha \end{aligned}$$

It then follows that:

$$\begin{aligned} x_0 &= \sqrt{x^2 + y^2}\cos(\alpha + \theta) \\ &= \sqrt{x^2 + y^2}\,[\cos\alpha\cos\theta - \sin\alpha\sin\theta] \\ &= x\cos\theta - y\sin\theta \end{aligned}$$

And that:

$$
\begin{aligned}
y_0 &= \sqrt{x^2+y^2}\sin(\alpha+\theta) \\
&= \sqrt{x^2+y^2}\,[\sin\alpha\cos\theta + \sin\theta\cos\alpha] \\
&= y\cos\theta + x\sin\theta \\
&= x\sin\theta + y\cos\theta
\end{aligned}
$$

With our extra 1, we claim that there is a linear tranformation f with:

$$
f\left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}\right) = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ 1 \end{bmatrix}
$$

If such a linear transformation exists then it can be represented by a matrix R and that matrix is dictated by what it does to the standard basis. More specifically if such an R exists then it would equal:

$$
\begin{bmatrix} f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) & f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) & f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \end{bmatrix}
$$

From above, our rotation would rotate $(1,0)$ to $(\cos\theta, \sin\theta)$, $(0,1)$ to $(-\sin\theta, \cos\theta)$, and $(0,0)$ to $(0,0)$. In other words:

$$
\begin{aligned}
f\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} \cos\theta \\ \sin\theta \\ 1 \end{bmatrix} \\
f\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 1 \end{bmatrix} \\
f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

It follows then by linearity we must have:

$$
\begin{aligned}
f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) &= f\left(\begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix}\right)\\
&= f\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) - f\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\\
&= \begin{bmatrix}\cos\theta\\ \sin\theta\\ 1\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix}\\
&= \begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
f\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) &= f\left(\begin{bmatrix}0\\1\\1\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix}\right)\\
&= f\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) - f\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\\
&= \begin{bmatrix}-\sin\theta\\ \cos\theta\\ 1\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix}\\
&= \begin{bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{bmatrix}
\end{aligned}
$$

Therefore if such a linear transformation f exists it must be represented by the matrix:

$$
R = \begin{bmatrix} f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) & f\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) & f\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}
$$

So we check that this R does in fact work:

$$
R\begin{bmatrix}x\\y\\1\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\1\end{bmatrix} = \begin{bmatrix}x\cos\theta - y\sin\theta\\ x\sin\theta + y\cos\theta\\ 1\end{bmatrix}
$$

Therefore we have a matrix that does the desired job and so in general for any θ we write:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2.3. Rotation around the origin by $\pi/6$ radians is given by the matrix

$$R(\pi/6) = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) & 0 \\ \sin(\pi/6) & \cos(\pi/6) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To rotate the point $(5, 3)$ we do:

$$\begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} (5\sqrt{3}-3)/2 \\ (5+3\sqrt{3})/2 \\ 1 \end{bmatrix}$$

to get the point $\big((5\sqrt{3}-3)/2, (5+3\sqrt{3})/2\big)$.

2.4 Combining Translations and Rotations

Now that we have matrices for translations and matrices for rotations we can combine these to get matrices for other transformations.

Example 2.4. Suppose we wish to first translate by -2 units in the x-direction and by 7 units in the y-direction and then rotate by $\pi/2$ radians.

The translation is:

$$T(-2, 7) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}$$

while the rotation is:

$$R(\pi/2) = \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) & 0 \\ \sin(\pi/2) & \cos(\pi/2) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore the matrix which does the translation and then the rotation will be the following:

$$R(\pi/2)T(-2,7) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -7 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that the translation, which happens first, goes on the right, because when we apply the combination to a point such as $(5,3)$ we would do:

$$\begin{aligned} R(\pi/2)T(-2,7) \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & -7 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix} \end{aligned}$$

so that the matrix multiplication for the translation happens before the matrix multiplication for the rotation.

Example 2.5. The classic example is rotation around a point other than the origin. Suppose we wish to rotate around the point $(4,7)$ by $\pi/6$ radians What we'll do is first shift the plane 4 units left and 7 units down, thereby placing our desired rotation point at the origin, rotate by $\pi/6$, and then shift back.

So we wish to do $T(-4,-7)$ then $R(\pi/6)$ then $T(4,7)$. In other words, the following; note the order because of which one we wish to do first:

$$T(4,7)R(\pi/6)T(-4,-7)$$

which is

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) & 0 \\ \sin(\pi/6) & \cos(\pi/6) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix}$$

or

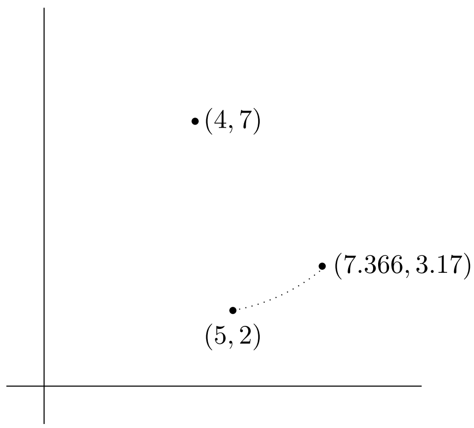
$$\begin{bmatrix} \sqrt{3}/2 & -1/2 & 15/2 - 2\sqrt{3} \\ 1/2 & \sqrt{3}/2 & 5 - 7\sqrt{3}/2 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 0.866 & -0.500 & 4.036 \\ 0.500 & 0.866 & -1.062 \\ 0 & 0 & 1.000 \end{bmatrix}$$

To summarize this new matrix rotates the plane around the point $(4, 7)$ by $\pi/6$ radians.

Then for example if we rotate the point $(5, 2)$ we get:

$$\begin{bmatrix} \sqrt{3}/2 & -1/2 & 15/2 - 2\sqrt{3} \\ 1/2 & \sqrt{3}/2 & 5 - 7\sqrt{3}/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.866 & -0.500 & 4.036 \\ 0.500 & 0.866 & -1.062 \\ 0 & 0 & 1.000 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7.366 \\ 3.17 \\ 1 \end{bmatrix}$$

We can see that this works:



2.5 Moving to 3D

We'll avoid giving too much detail here since it should be fairly clear at this point where all of this comes from.

In analogy to two dimensions a point in 3D will be represented by a vector with an additional entry:

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

A translation by a, b, c units in the x, y, z directions respectively will then be given by the matrix

$$T(a, b, c) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation is a tricky business because we now have to rotate around axes and an axis could be any line. However it's fairly easy to write down rotations around the x, y and z axes in the direction dictated by the right-hand rule in the sense that following the rotation with the fingers of the right hand points the thumb in the positive axis direction.

In general when we talk about rotation around an arbitrary axis we'll make sure that the axis has direction and that the rotation obeys the right-hand rule with regards to this direction.

Rotation around the z-axis is easiest because it comes from the two-dimensional case where we're moving x and y but not z. The following matrix leaves the z value alone and rotates the positive x-axis toward the positive y-axis thereby obeying our right-hand-rule wishes:

$$RZ(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If we simply swap some positions we get rotation around the x-axis. The following matrix leaves the x value alone and rotates the positive y-axis toward the positive z-axis thereby obeying our right-hand-rule wishes:

$$RX(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

When we rotate around the y-axis we need to be careful. Obeying the right-hand-rule wishes for the y-axis insists that the positive z-axis should rotate toward the positive x-axis.

This means when we adapt $RZ(\theta)$ we need to interchange the x and z positions, yielding:

$$RY(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 2.6. To rotate around the y-axis by $7\pi/6$ radians we use the matrix:

$$RY(7\pi/6) = \begin{bmatrix} \cos(7\pi/6) & 0 & \sin(7\pi/6) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(7\pi/6) & 0 & \cos(7\pi/6) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & -\sqrt{3}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

At this point just like in 2D we can combine translations and rotations by simply multiplying matrices.

To rotate about an axis which is parallel to either the x, y or z axis we simply translate, rotate, then translate back, as in 2D.

Example 2.7. To rotate around the axis given by the line $x = 1$, $y = 4$ with upwards direction by $\pi/4$ radians we first shift $T(-1, -4, 0)$, then do $RZ(\pi/4)$, then do $T(1, 4, 0)$:

$$\begin{aligned} &T(1,4,0)RZ(\pi/4)T(-1,-4,0) \\ &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) & 0 & 0 \\ \sin(\pi/4) & \cos(\pi/4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 3\sqrt{2}/2 + 1 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 4 - 5\sqrt{2}/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Example 2.8. To rotate around the axis given by the line $y = 2$, $z = -1$ with orientation opposite to the x-axis by $3\pi/4$ radians we first shift $T(0, -2, 1)$, then do $RX(-3\pi/4)$, then do $T(0, 2, -1)$:

$$T(0,2,-1)RX(-3\pi/4)T(0,-2,1)$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 & 3\sqrt{2}/2 + 2 \\ 0 & -\sqrt{2}/2 & -\sqrt{2}/2 & \sqrt{2}/2 - 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

To rotate about axes which are not parallel to one of the three main axes is trickier and is explored in the exercises.

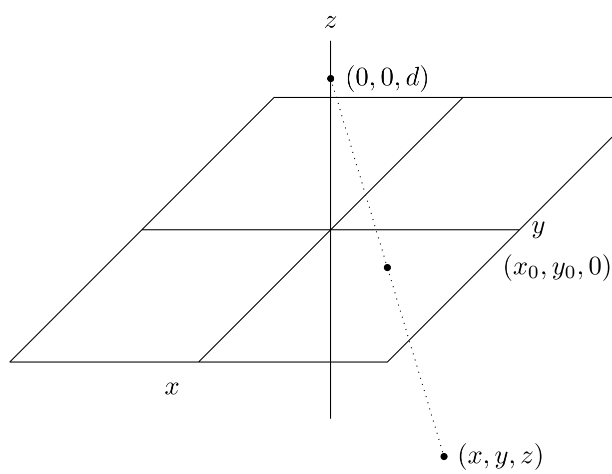
2.6 Perspective Projection in 3D

The last issue we'd like to address is the fact that computer graphics, while composed of 3D points, are rendered in 2D on your computer screen. This is done with a sense of perspective, meaning that things that are farther away look smaller than things that are closer.

Can we accomplish this with a matrix multiplication? The short answer is no, however there is an easy fix, and we'll see why and how as we build a simple example.

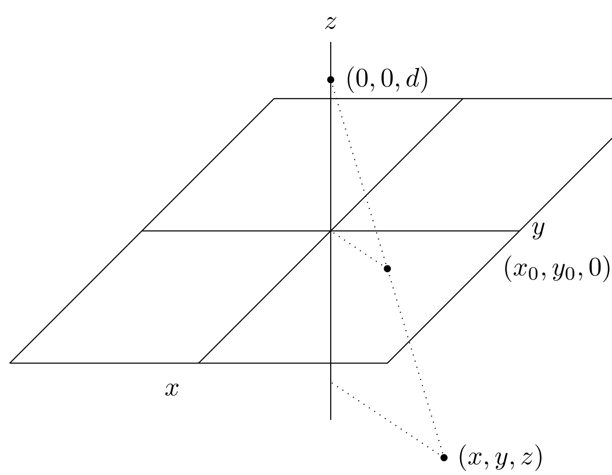
2.6.1 Perspective Projection from $z = d > 0$

For our example we'll assume that the *center of perspective* (in simple terms, the viewpoint) is positioned at a position $z = d > 0$ along the positive z-axis, that the object lies below the xy-plane, and that we wish to project our points to the xy-plane as illustrated by the following picture:



The goal is to find $(x_0, y_0, 0)$ in terms of (x, y, z).

To see what this should be observe that there are two similar right triangles in the above picture:



These similar triangles tell us that:

$$\frac{d}{\sqrt{x_0^2 + y_0^2}} = \frac{d - z}{\sqrt{x^2 + y^2}}$$

Next, if θ is the polar angle for both (x, y, z) and $(x_0, y_0, 0)$ (these have the same polar angle) then we know $x = \sqrt{x^2 + y^2} \cos\theta$ and $x_0 = \sqrt{x_0^2 + y_0^2} \cos\theta$ and so the above equation becomes:

$$
\begin{aligned}
\frac{d}{\sqrt{x_0^2+y_0^2}} &= \frac{d-z}{\sqrt{x^2+y^2}} \\
\frac{d}{x_0/\cos\theta} &= \frac{d-z}{x/\cos\theta} \\
\frac{d}{x_0} &= \frac{d-z}{x} \\
x_0 &= \frac{dx}{d-z} \\
x_0 &= \frac{x}{1-z/d}
\end{aligned}
$$

A similar argument with y and sin yields

$$y_0 = \frac{y}{1-z/d}$$

In summary we know that

$$(x_0, y_0, 0) = \left(\frac{x}{1-z/d}, \frac{y}{1-z/d}, 0\right)$$

So if this is going to work with the way we are representing points we need:

$$f\left(\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{x}{1-z/d} \\ \frac{y}{1-z/d} \\ 0 \\ 1 \end{bmatrix}$$

If such a linear transformation exists then it can be represented by a matrix M and that matrix is dictated by what it does to the standard basis. More specifically if such an M exists then:

$$M = \left[f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) \quad f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) \quad f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) \quad f\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) \right]$$

In order to find these, note that f being linear gives us the following:

$$f\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)=f\left(\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right)-f\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\0\\0\\1\end{bmatrix}-\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$f\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)=f\left(\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right)-f\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\\0\\1\end{bmatrix}-\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$$f\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right)=f\left(\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right)-f\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}-\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

So that if such an M exists then:

$$M=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}$$

However this matrix does not do what we need f to do:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\\1\end{bmatrix}=\begin{bmatrix}x\\y\\0\\1\end{bmatrix}\neq\begin{bmatrix}\frac{x}{1-z/d}\\\frac{y}{1-z/d}\\0\\1\end{bmatrix}$$

and so f is not linear.

So what are we to do? The answer is related to our fourth coordinate which has, up until this point, always been a 1.

Instead of directly trying to send:

$$\begin{bmatrix}x\\y\\z\\1\end{bmatrix}\mapsto\begin{bmatrix}\frac{x}{1-z/d}\\\frac{y}{1-z/d}\\0\\1\end{bmatrix}$$

Consider this matrix product:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/d & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ 1 - z/d \end{bmatrix}$$

The result of this product is a multiple of the desired result. If we divide the result by $1 - z/d$ we get:

$$\begin{bmatrix} x \\ y \\ 0 \\ 1 - z/d \end{bmatrix} \div (1 - z/d) = \begin{bmatrix} \frac{x}{1-z/d} \\ \frac{y}{1-z/d} \\ 0 \\ 1 \end{bmatrix}$$

as desired.

The solution therefore is to define the projection matrix as follows:

$$P(d) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/d & 1 \end{bmatrix}$$

Then after we project we divide each point by its fourth entry to fix the x and y coordinates.

While this final process is not linear this is not an issue and since it only needs to be done once, after all the other transformations have happened, it is not that much of a hassle.

Example 2.9. Consider the $6 \times 6 \times 6$ cube with vertices:

$$(-3,-3,-3), (3,-3,-3), (3,3,-3), (-3,3,-3), (-3,-3,-9), (3,-3,-9), (3,3,-9), (-3,3,-9)$$

If we treat these as vectors we can put them all together in a matrix:

$$A = \begin{bmatrix} -3 & -3 & -3 & -3 & 3 & 3 & 3 & 3 \\ -3 & -3 & 3 & 3 & -3 & -3 & 3 & 3 \\ -3 & -9 & -3 & -9 & -3 & -9 & -3 & -9 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

we can project them all at once. If $d = 10$ then we get:

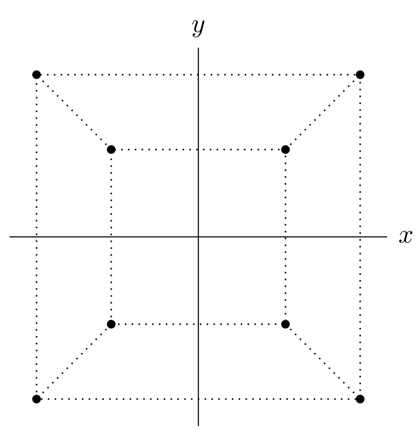
$$P(10)A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/10 & 1 \end{bmatrix} \begin{bmatrix} -3 & -3 & -3 & -3 & 3 & 3 & 3 & 3 \\ -3 & -3 & 3 & 3 & -3 & -3 & 3 & 3 \\ -3 & -9 & -3 & -9 & -3 & -9 & -3 & -9 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3.00 & -3.00 & -3.00 & -3.00 & 3.00 & 3.00 & 3.00 & 3.00 \\ -3.00 & -3.00 & 3.00 & 3.00 & -3.00 & -3.00 & 3.00 & 3.00 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.30 & 1.90 & 1.30 & 1.90 & 1.30 & 1.90 & 1.30 & 1.90 \end{bmatrix}$$

Each of these columns is then individually scaled so that the bottom entry is 1. Here is the result approximated:

$$\begin{bmatrix} -2.308 & -1.579 & -2.308 & -1.579 & 2.308 & 1.579 & 2.308 & 1.579 \\ -2.308 & -1.579 & 2.308 & 1.579 & -2.308 & -1.579 & 2.308 & 1.579 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.000 & 1.000 & 1.000 & 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \end{bmatrix}$$

We then extract the top three vectors as our new point. If we plot these in the xy plane we see the result. Here I've connected the corners by lines for added effect:



At this point we can do really fancy stuff.

Example 2.10. Suppose we wanted to take our cube from above, rotate it around the x-axis by $\pi/6$, the y-axis by $\pi/3$ and the z-axis by $\pi/12$, then shift it in the z-direction by -1, then project it with $d = 10$.

The five necessary matrices are:

$$RX(\pi/6) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 & 0 \\ 0 & 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$RY(\pi/3) = \begin{bmatrix} 1/2 & 0 & -\sqrt{3}/2 & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{3}/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$RZ(\pi/12) = \begin{bmatrix} (\sqrt{2}+\sqrt{6})/4 & (\sqrt{2}-\sqrt{6})/4 & 0 & 0 \\ (\sqrt{6}-\sqrt{2})/4 & (\sqrt{2}+\sqrt{6})/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T(0,0,-1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P(10) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/10 & 1 \end{bmatrix}$$

The result of doing each of these in the correct order is:

$$P(10)T(0,0,-1)RZ(\pi/12)RY(\pi/3)RX(\pi/6) \approx \begin{bmatrix} 0.4830 & 0.1941 & 0.8539 & 0 \\ 0.1294 & 0.9486 & -0.2888 & 0 \\ 0 & 0 & 0 & 0 \\ 0.0866 & -0.0250 & -0.0433 & 1.1000 \end{bmatrix}$$

And applied to the corners of the cube:

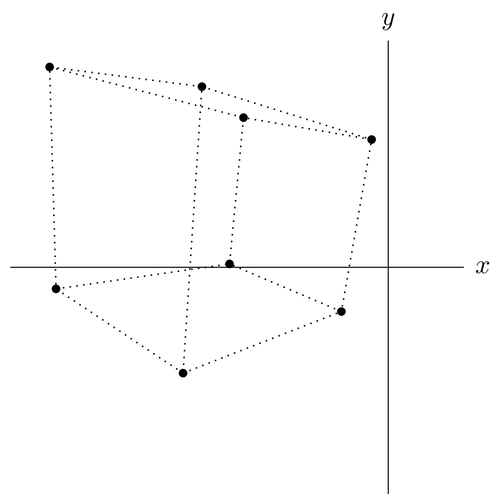
$$P(10)T(0,0,-1)RZ(\pi/12)RY(\pi/3)RX(\pi/6) \begin{bmatrix} -3 & -3 & -3 & -3 & 3 & 3 & 3 & 3 \\ -3 & -3 & 3 & 3 & -3 & -3 & 3 & 3 \\ -3 & -9 & -3 & -9 & -3 & -9 & -3 & -9 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} -4.5928 & -9.7159 & -3.4281 & -8.5512 & -1.6950 & -6.8181 & -0.5303 & -5.6535 \\ -2.3674 & -0.6344 & 3.3241 & 5.0572 & -1.5910 & 0.1421 & 4.1005 & 5.8336 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.8451 & 1.1049 & 0.6951 & 0.9549 & 1.3647 & 1.6245 & 1.2147 & 1.4745 \end{bmatrix}$$

and then scaled:

$$\begin{bmatrix} -5.4346 & -8.7935 & -4.9318 & -8.9551 & -1.2420 & -4.1970 & -0.4366 & -3.8341 \\ -2.8014 & -0.5741 & 4.7822 & 5.2960 & -1.1658 & 0.0875 & 3.3757 & 3.9563 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \end{bmatrix}$$

Here's the picture with lines added for clarity:



2.6.2 Perspective Projection from Other Places

In order to get a perspective from another location we need to specify a few things, namely the plane on which we're projecting, a perpendicular axis on which the center of perspective lies, and how far away that center of perspective is. We also need to know which way is "up" for the view. That "up" direction will be given by a vector in the plane.

We then find the transformation which moves that plane to the xy-plane, puts the axis along the positive z-axis, and rotates around the positive z-axis so that the "up" direction points in the positive y-direction. We then find out how far the center of perspective is from the plane and use that for d.

Then use the resulting transformation to move the objects and lastly we do as in the previous section.

Notice that the result lies in the xy-plane and we don't need to move it all back, because getting it into the xy-plane is the end result for viewing.

This is calculation-heavy and sensitive and is explored in the exercises.

2.7 Matlab

Here are all the matrices in Matlab, along with necessary `syms`.

```
% 2D Stuff
syms TT(a,b) R(t)
TT(a,b)=[1 0 a;0 1 b;0 0 1];
R(t)=[cos(t) -sin(t) 0;sin(t) cos(t) 0;0 0 1];
% 3D Stuff
syms T(a,b,c) RX(t) RY(t) RZ(t) P(d)
T(a,b,c)=[1 0 0 a;0 1 0 b;0 0 1 c;0 0 0 1];
RX(t)=[1 0 0 0 ; 0 cos(t) -sin(t) 0;0 sin(t) cos(t) 0 ; 0 0 0 1];
RY(t)=[cos(t) 0 sin(t) 0;0 1 0 0;-sin(t) 0 cos(t) 0 ;0 0 0 1];
RZ(t)=[cos(t) -sin(t) 0 0;sin(t) cos(t) 0 0 ; 0 0 1 0; 0 0 0 1];
P(d)=[1 0 0 0;0 1 0 0;0 0 0 0;0 0 -1/d 1];
```

We enter a set of points as the transpose of a vector only because it makes it easier to enter the points one by one:

```
>> P=transpose([1,2,1;-3,0,1])
P =
     1    -3
     2     0
     1     1
```

It's then easy to apply transformations to points:

```
>> P=transpose([1,2,1;-3,0,1]);
>> NEWP=TT(4,5)*R(pi/6)*P
NEWP =
[  3^(1/2)/2 + 3, 4 - (3*3^(1/2))/2]
[ 3^(1/2) + 11/2,               7/2]
[              1,                 1]
```

If you would prefer approximations because they're easier to read, simply wrap the result in `vpa` which stands for *variable precision arithmetic*, and give it the number of digits as a second argument:

```
>> P=transpose([1,2,1;-3,0,1]);
>> NEWP=vpa(TT(4,5)*R(pi/6)*P,4)
NEWP =
[ 3.866, 1.402]
[ 7.232,   3.5]
[   1.0,   1.0]
```

We can plot these in 2D (plot not shown here) using the following, which takes out the x and y-coordinates and plots them against one another:

```
>> scatter(NEWP(1,:),NEWP(2,:))
```

In 3D, when we need to post-process the final matrix so that the fourth row is all 1s, we can do this easily. Here is the full process where we define two points, apply some rotations and a projection, and then post-process and plot. In addition the final two command set the x and y-axes to be proportionally sized and set the ranges of those axes.

```
>> PTS=transpose([1 2 3 0;0 -1 3 2]);
>> NEWPTS=P(10)*RY(pi/4)*RZ(pi/4)*PTS;
>> for i=1:size(NEWPTS,2);
NEWPTS(:,i)=NEWPTS(:,i)/NEWPTS(4,i);
end;
>> scatter(NEWPTS(1,:),NEWPTS(2,:),'filled')
>> axis square
>> axis([-5 5 -5 5])
```

2.8 Exercises

Exercise 2.1. In two dimensions write down the translation matrix which shifts $+8$ in the x-direction and -3 in the y-direction. Apply this matrix to the points $(1,2)$ and $(0,-10)$.

Exercise 2.2. In three dimensions write down the translation matrix which shifts $+1$ in the x-direction, 2 in the y-direction and -7 in the y-direction. Apply this matrix to the points $(1,2,0)$ and $(1,4,-3)$.

Exercise 2.3. In two dimensions write down the rotation matrix which rotates around the origin by $7\pi/6$ radians. Apply this matrix to the points $(1,2)$ and $(0,-10)$.

Exercise 2.4. In two dimensions write down the rotation matrix which rotates around the origin by $2\pi/3$ radians clockwise. Apply this matrix to the points $(0,3)$ and $(1,-1)$.

Exercise 2.5. In two dimensions write down the matrix which rotates around the origin by $\pi/6$ radians and then translates by -3 in the x-direction and 5 in the y-direction. Apply this matrix to the points $(0,3)$ and $(1,-1)$.

Exercise 2.6. In two dimensions write down the matrix which translates by -3 in the x-direction and 5 in the y-direction and then rotates around the origin by $\pi/6$ radians. Apply this matrix to the points $(0, 3)$ and $(1, -1)$.

Exercise 2.7. In two dimensions find the rotation matrix which will rotate around the point $(-3, 2)$ by an angle of $\pi/4$ radians. Apply this matrix to the points $(4, 5)$ and $(0, 0)$.

Exercise 2.8. In two dimensions find the rotation matrix which will rotate around the point $(6, -3)$ by an angle of $7\pi/6$ radians. Apply this matrix to the points $(-2, 1)$ and $(0, 0)$.

Exercise 2.9. In two dimensions find the image of the three points

$$(6, 3), (4, 1), (7, 1)$$

under rotation around the point $(8, 2)$ by $\pi/3$ radians. Sketch the original points and the images.

Exercise 2.10. In two dimensions find the image of the three points

$$(6, 3), (4, 1), (7, 1)$$

under rotation around the point $(0, 7)$ by $\pi/4$ radians. Sketch the original points and the images.

Exercise 2.11. In two dimensions write down the matrix (simplified) for rotation around the point (a, b) by θ radians.

Exercise 2.12. In two dimensions find the matrix transformation composed of one translation followed by one rotation which moves the line segment joining $(4, 3)$ to $(7, 2)$ so that the point $(4, 3)$ moves to the origin and the segment lies along the positive x-axis. Then find the image of the origin under this transformation.

Exercise 2.13. In two dimensions find the matrix transformation composed of one translation followed by one rotation which moves the line segment joining $(4, 3)$ to $(7, 2)$ so that the point $(7, 2)$ moves to the origin and the segment lies along the positive y-axis. Then find the image of the point $(-2, 1)$ under this transformation.

Exercise 2.14. Reflection through a line through the origin followed by reflection through another line through the origin results in a rotation around the origin. Show this as follows:

(a) Write down the matrix which reflects in the x-axis.

(b) Use this to find the matrix which reflects in the line which makes an angle of θ degrees with the positive x-axis.

(c) Compose two of these, simplify, and explain why the result is a rotation around the origin.

Exercise 2.15. In three dimensions find the image of the three points

$$(1, 2, 3), (-2, 3, 1), (3, 2, 2)$$

under rotation around the y-axis by $\pi/4$.

Exercise 2.16. In three dimensions find the image of the three points

$$(0, 3, 1), (-6, 3, 1), (3, 1, 10)$$

under shifting by $(+2, -3, -6)$ followed by rotation around the z-axis by $7\pi/6$.

Exercise 2.17. In three dimensions find the image of the three points

$$(1, 2, 3), (-2, 3, 1), (3, 2, 2)$$

under the perspective projection with center of perspective at $z = 10$.

Exercise 2.18. In three dimensions find the image of the three points

$$(4, 3, 1), (-3, 4, 2), (5, -1, 6)$$

under the perspective projection with center of perspective at $z = 20$.

Exercise 2.19. In three dimensions we can rotate around an axis parallel to the x-axis by translating the desired axis so that it's on top of the x-axis, rotating, and then translating back. Using this method find the rotation matrix which will rotate around the line $y = 2$, $z = 5$ with direction identical to the x-axis by $3\pi/4$ radians.

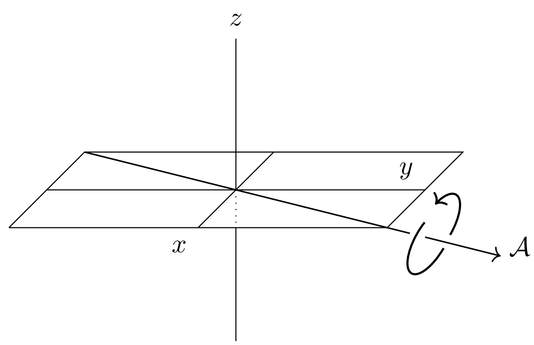
Exercise 2.20. In three dimensions we can rotate around an axis parallel to the y-axis by translating the desired axis so that it's on top of the y-axis, rotating, and then translating back. Using this method find the rotation matrix which will rotate around the line $x = -2$, $z = 4$ with direction opposite to the y-axis by $2\pi/3$ radians.

Exercise 2.21. In three dimensions we can rotate around an axis parallel to the z-axis by translating the desired axis so that it's on top of the z-axis, rotating, and then translating back. Using this method find the rotation matrix which will rotate around the line $x = 1$, $y = -5$ with direction identical to the z-axis by $4\pi/3$ radians.

Exercise 2.22. Consider the axis $\mathcal{A}$ lying along the vector

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

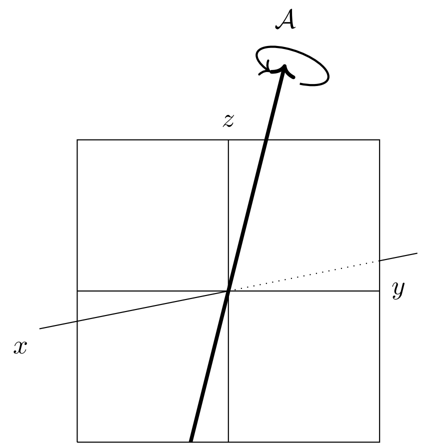
By rotating around the z-axis by an appropriate amount this axis can be placed on top of the x-axis. Use this fact to find the rotation matrix which will rotate by an angle of $\pi/6$ around $\mathcal{A}$.



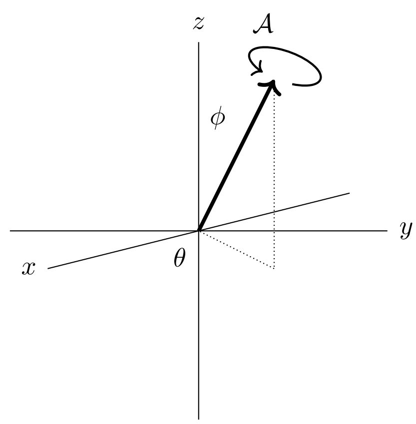
Exercise 2.23. Consider the axis $\mathcal{A}$ lying along the vector

$$\begin{bmatrix} 0 \\ 1 \\ \sqrt{3} \end{bmatrix}$$

By rotating around the x-axis by an appropriate amount this axis can be placed on top of the z-axis. Use this fact to find the rotation matrix which will rotate by an angle of $\pi/3$ around $\mathcal{A}$.



Exercise 2.24. If a desired axis of rotation $\mathcal{A}$ is given in terms of the spherical coordinate angles ϕ and θ:



Then rotation around $\mathcal{A}$ by α radians can be easily obtained by first rotating around the z-axis by $-\theta$ radians (at this point the desired axis is in the xz-plane), then rotating around the y-axis by $-\phi$ radians (at this point the desired axis is on top of the positive z-axis). We then rotate around the z-axis by α radians. Finally we undo the first two rotations.

Calculate (and simplify) the matrix for the rotation if $\phi = \pi/6$, $\theta = \pi/3$ and $\alpha = 3\pi/4$.

Exercise 2.25. Projection onto one of the other two coordinate planes (the xz-plane and the yz-plane) can easily be accomplished by rotation. For example if we wish to project onto the xz-plane what we do is first rotate the y-axis to the z-axis (which is a rotation around the x-axis), then project, then rotate back.

(a) Write down the projection matrix which does this.

(b) Use this to project the three points

$$(1, 2, 3), (4, -1, 0), (5, 2, 3)$$

with center of perspective at $y = 10$.

Exercise 2.26. Projection onto one of the other two coordinate planes (the xz-plane and the yz-plane) can easily be accomplished by rotation. For example if we wish to project onto the yz-plane what we do is first rotate the x-axis to the z-axis (which is a rotation around the y-axis), then project, then rotate back.

(a) Write down the projection matrix which does this.

(b) Use this to project the three points

$$(1, 2, 3), (4, -1, 0), (5, 2, 3)$$

with center of perspective at $x = 10$.

Exercise 2.27. Design/construct a three-dimensional object using points. Store those points in a $4 \times k$ matrix and then apply a series of interesting transformations. Find the resulting points and plot. In your submission plot the points before the transformations, give the series of transformations both by description and by matrix and give the overall resulting matrix. Finally plot the points after the transformations.

Exercise 2.28. As the center of perspective moves along the z-axis to infinity the projection matrix becomes $P_\infty = \lim_{d \to \infty} \bar{P}_d$.

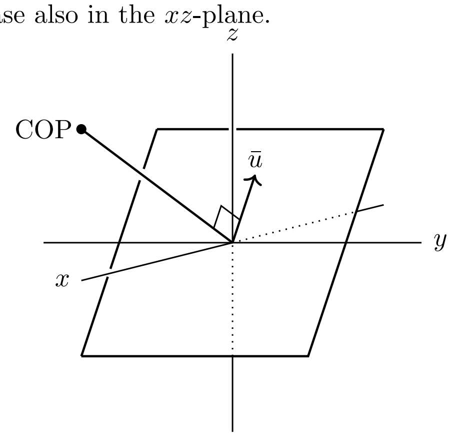
(a) Calculate this matrix.

(b) Find

$$P_\infty \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

and explain why it makes sense from a geometric viewpoint. Pictures will probably help.

Exercise 2.29. Suppose we wish our view to work as in the following picture. In this picture assume the COP is at $(5, 0, 3)$, the plane which is visible (and which is perpendicular to the line joining the COP to the origin) is the plane we wish to act as the xy-plane, the vector $\bar{u}$ is the vector which indicates which way is up, meaning which needs to act as the y-axis. This vector lies in the plane and in this case also in the xz-plane.



Our goal is to construct the projection matrix which projects onto this plane as if it were the xy-plane. Do this using the following steps:

(a) Write down the matrix which rotates around the y-axis an appropriate amount so that the COP lies on the positive z-axis and the plane lies on the xy-plane.

(b) Assuming this has been done, write down the matrix which rotates around the z-axis an appropriate amount so that $\bar{u}$ lies on the positive y-axis.

(c) Multiply these two in the correct order followed by the appropriate $P(d)$ to finish the job.

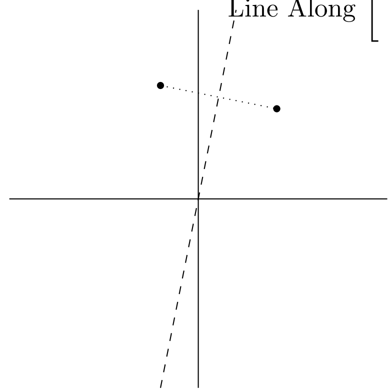
(d) Apply the matrix to the Matlab box, post-process and plot. Print and attach.

Exercise 2.30. Use the approach of the proceeding problem to find the projection matrix corresponding to the viewpoint where the COP is at $(10, 10, 10)$ looking directly toward the origin. The projection plane contains the origin and the "up" direction is given by the vector $[2, -1, -1]$.

Exercise 2.31. In two dimensions find the generic rotation matrix which will rotate by an angle of θ around the point (x_0, y_0).

Exercise 2.32. In three dimensions suppose we projected to the plane $z = z_0$ instead of the plane $z = 0$. To which coordinates would the point (x_0, y_0, z_0) be sent? Justify.

Exercise 2.33. In two dimensions find the matrix which would reflect the xy-plane in the line through the origin with direction vector $\begin{bmatrix} a \\ b \end{bmatrix}$.



Exercise 2.34. Prove that if points in the xy-plane are treated simply as vectors

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

that the following are not linear:

(a) Rotation around a point which is not the origin.

(b) Translation.

Exercise 2.35. Prove that the matrices representing the following pairs of transformations are inverses of each other:

(a) $T(a, b)$ and $T(-a, -b)$.

(b) $R(\theta)$ and $R(-\theta)$.

Exercise 2.36. Prove that $RX(\pi)RY(\pi)RZ(\pi) = I$. What interesting inverse facts can you extract from this?

Exercise 2.37. Show (in 2D and in 3D) using the corresponding matrices that a translation followed by a translation equals a translation.

Exercise 2.38. Is it true or false that for all a, b, θ we have the following. Provide evidence.

$$R(\theta)T(a, b) = T(a, b)R(\theta)$$

Exercise 2.39. Write down the 3×3 matrix which reflects the plane through the origin, sending each point to its opposite. Then use this to find the matrix which reflects the plane through the point (x_0, y_0).

Exercise 2.40. It's possible to do a 2D version of projection, where projection is done with the COP at $y = d$ and projection is onto the x-axis. Develop this. Specifically, what would the projection matrix look like, how would it work, would post-processing be necessary and so on?