# Game Theory

We will apply linear programming to game theory.

The basic idea comes from John von Neumann, the twentieth century mathematician who developed the von Neumann architecture of computers, among other things.

We are going to consider matrix games. Two player zero sum. You lose then he wins. You win then he loses.

## Rock paper scissors

This is the type of game we will study.

## Prisoner's Dilemma

Two prisoners are accused of murder.

They were both caught with an illegal firearm.

They are interrogated separately and given a choice.

Confess or else your partner will confess and go free.

If they both confess, they get 10 years in jail.

If only one confesses, he goes free and the other gets 20 years.

If neither confesses, they both get one year in jail for the weapons charge.

This one is not a zero-sum game. The total jail time is either 1+1=2 or 20.

## Chess

A game like chess has no immediate payoff.

It is deterministic.

It has a game tree.

## Definition: Two player zero-sum, Simultaneous Moves

A two player game is called zero sum if the payoffs for every outcome sum to zero. If player A wins 5 then player B loses 5.

The prisoner's dilemma is not zero sum because the payoffs total to -20 or to -2.

Rock-paper-scissors is zero sum because when one player wins, the other player loses. The other outcome is a tie which also sums to zero.

## Definition: Dominated Strategy

One choice can be clearly worse than another.

A more subtle situation involves when two choices have the same outcomes.

In either case, the game is simplified by removing redundant choices.

## Definition: Mixed strategy

A player employs a mixed strategy when he chooses between different strategies based on a probability.

If there are n-choices, then a mixed strategy consists of a probabiity vector of length n, i.e., the sum is 1 and the entries are between 0 and 1.

For example, in rock-paper-scissors, one mixed strategy is to choose rock 1/2 the time and paper 1/2 the time.

In places of payoffs, if one player employs a mixed strategy, one evaluates the strategy using probabilistic expectation.

For the sake of generality, one considers a fixed strategy, such as always choosing Rock, as a special case of a mixed strategy where the

## Payoff Matrix

The payoff matrix is an m by n matrix. The first player chooses a row, and the second player chooses a column. The choices are made simultaneously in principle. The outcome is the value of the matrix at the row and column. That is the amount that player Y pays player X. If the value is negative, that is favor of player Y.

## Convert the game theory to a system of linear equations

No matter what choice player P makes, he cannot beat the expectation.

But what is the expectation?

Assume we have removed dominated strategies.

It is $m_{ij} \cdot p_i \cdot q_j$

Imagine one player has fixed his strategy. The other player has a linear programming problem to solve.

After the dominated strategies are removed, choose a mixed strategy where the opponent

Assume we have removed dominated strategies.

$(m_{ij}\, q_j) \cdot p_i \leq$ expectation

But if the other player has his strategy, it is another linear programming problem.

$(m_{ij}\, p_i) \cdot q_j \geq$ expectation

This is exactly duality for linear programming.

One changes the role of the constraints and the decision variables.

Min for one is the max for the dual problem.

## Dual Linear Programming problems

Switch

LinearProgramming[c, m, b]

to

LinearProgramming[ -b, - m-transpose, - c]

e.g., minimize c dot x where m dot x ≤ b

to

maximize b dot x where m dot x ≥ c

Consider an example:

minimize x1 + 2 x2

subject to 3 x1 + 5 x2 ≥ 8

In[∘]:=
```
LinearProgramming[{1, 2}, {{3, 5}}, {8}]
```

Out[∘]= $\left\{\frac{8}{3}, 0\right\}$

In[∘]:= $\{1, 2\}.\left\{\frac{8}{3}, 0\right\}$

Out[∘]= $\frac{8}{3}$

Switching minimum to maximum entails changing the sign.  Transpose the constraints and switch c and b.

In[∘]:=
```
LinearProgramming[{-8}, {{-3}, {-5}}, {-1, -2}]
```

Out[∘]= $\left\{\frac{1}{3}\right\}$

In[∘]:= $\{-8\}.\left\{\frac{1}{3}\right\}$

Out[∘]= $-\frac{8}{3}$

## Problems

(Word problem)  Can the first player guarantee an expectation of 1/2?  What are the linear inequalities?

What are the dominated strategies in this game?

Solve this game.

How many solutions are there for this game?

If player A bluffs 10% of the time, what should player B do?