Leontief Input-Output Models

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1.1 Introduction

1.1.1 Introduction

Wassily Leontief was an economist who was one of the first people to do computational analysis of economics. Moreover his work involved one of the first uses of a computer to produce this analysis, done in 1949 at Harvard. For his work he won a Nobel Prize in 1973.

Leontief based his approach on the idea that an economy is basically divided into Sectors and each Sector produces a Product. In order to produce its Product each Sector requires input which must come from possibly all of the Sectors, including itself.

Consequently, overall, the amount that the full economy must produce has to include any desired external demand as well as the internal demand which feeds back into the economy so that each Sector can do its job.

As a pseudo-equation:

$$\text{Total Amount Produced} = \text{Internal Demand} + \text{External Demand}$$

To see this more clearly let's look at a basic example:

Example 1.1. Suppose there are three Sectors each producing units of its Product. In order for each Sector to function it needs some of its own Product as well as some of the other Sectors' Products. Suppose we have the following:

- For Sector 1 to produce 1 unit of Product 1 it takes 0.10 units of Product 1, 0.20 units of Product 2, and 0.25 units of Product 3.
- For Sector 2 to produce 1 unit of Product 2 it takes 0.15 units of Product 1, 0 units of Product 2, and 0.40 units of Product 3.
- For Sector 3 to produce 1 unit of Product 3 it takes 0.12 units of Product 1, 0.30 units of Product 2, and 0.20 units of Product 3.

If the goal is for the Sectors to produce p_1 units of Product 1, p_2 units of Product 2, and p_3 units of Product 3, then what is the total internal requirement? This is the internal demand.

Consider for example how much Product 1 is required in total:

- To produce p_1 units of Product 1 requires $0.10p_1$ units of Product 1.
- To produce p_2 units of Product 2 requires $0.15p_2$ units of Product 1.
- To produce p_3 units of Product 3 requires $0.12p_3$ units of Product 1.

Therefore in total we require $0.10p_1 + 0.15p_2 + 0.12p_3$ units of Product 1.

Following this same approach we find that in total:

- We will need $0.10p_1 + 0.15p_2 + 0.12p_3$ units of Product 1.
- We will need $0.20p_1 + 0p_2 + 0.30p_3$ units of Product 2.
- We will need $0.25p_1 + 0.40p_2 + 0.20p_3$ units of Product 3.

Since we must produce these quantities to satisfy internal demand we therefore have:

$$\begin{aligned}
p_1 &= 0.10p_1 + 0.15p_2 + 0.12p_3 \\
p_2 &= 0.20p_1 + 0p_2 + 0.30p_3 \\
p_3 &= 0.25p_1 + 0.40p_2 + 0.20p_3
\end{aligned}$$

This may also be written as:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = p_1 \begin{bmatrix} 0.10 \\ 0.20 \\ 0.25 \end{bmatrix} + p_2 \begin{bmatrix} 0.15 \\ 0 \\ 0.40 \end{bmatrix} + p_3 \begin{bmatrix} 0.12 \\ 0.30 \\ 0.20 \end{bmatrix}$$

which can be written nicely in matrix form as:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0.10 & 0.15 & 0.12 \\ 0.20 & 0 & 0.30 \\ 0.25 & 0.40 & 0.20 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

In addition suppose there is an external demand of d_1, d_2 and d_3 units for the three Products respectively then the total that needs to be produced is:

$$\underbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}}_{\text{Total Produced}} = \underbrace{\begin{bmatrix} 0.10 & 0.15 & 0.12 \\ 0.20 & 0 & 0.30 \\ 0.25 & 0.40 & 0.20 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}}_{\text{Internal Demand}} + \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{\text{External Demand}}$$

1.1.2 Generalization

Definition 1.1.2.1. The *Leontief Input-Output Model* is given by:

$$\bar{p} = M\bar{p} + \bar{d}$$

Definition 1.1.2.2. The matrix M is the *consumption matrix.*

Definition 1.1.2.3. The consumption matrix is made up of *consumption vectors.* The j^{th} column is the j^{th} consumption vector and contains the necessary input required from each of the Sectors for Sector j to produce one unit of output.

Notice that in the consumption matrix the requirements for producing one unit of a given Product becomes a column (rather than a row) of the matrix. This is often a source of confusion.

Definition 1.1.2.4. The vector $\bar{p}$ is the *production vector*.

Definition 1.1.2.5. The vector $\bar{d}$ is the *external demand vector*.

Definition 1.1.2.6. The vector $M\bar{p}$ is the *internal demand vector*.

Two associated definitions:

Definition 1.1.2.7. An economy is *open* if $\bar{d} \neq \bar{0}$ and *closed* if $\bar{d} = \bar{0}$.

In a closed economy all of the output that is produced by the various Sectors is fed back in as input to those Sectors - there is no external demand. If the economy is closed this has serious ramifications on M which will be discussed later. Closed economies are mathematically rare.

The terms *open* and *closed* are used in other ways in economics as well so be cautious.

1.1.3 Goal

The primary goal here is the following: We know the consumption matrix and the external demand and we wish to set the amounts that each Sector must produce in order to satisfy both internal and external demand.

In other words we know M and $\bar{d}$ and we wish to know $\bar{p}$.

1.2 Solving Problems

If we blindly attempted to solve for $\bar{p}$ we might try:

$$\begin{aligned} M\bar{p} + \bar{d} &= \bar{p} \\ \bar{p} - M\bar{p} &= \bar{d} \\ (I - M)\bar{p} &= \bar{d} \end{aligned}$$

However at this point we make a few mathematical observations:

- If $I - M$ is invertible this has only one solution which is $\bar{p} = (I - M)^{-1}\bar{d}$.
- If $I - M$ is invertible and $\bar{d} = \bar{0}$ then the only solution is $\bar{p} = \bar{0}$.
- If $I - M$ is not invertible then this may have none or infinitely many solutions.
- If $I-M$ is not invertible and $\bar{d} = \bar{0}$ then there are infinitely many solutions.

1.2.1 Method - Invertible $I - M$

Let's first explore the case where $I - M$ is invertible. This happens most of the time because most matrices are inversible, statistically speaking, since most matrices have nonzero determinant. In this case we can simply solve as noted above:

$$\bar{p} = (I - M)^{-1}\bar{d}$$

Example 1.2. Suppose we have our initial example - three Sectors with consumption matrix:

$$M = \begin{bmatrix} 0.10 & 0.15 & 0.12 \\ 0.20 & 0 & 0.30 \\ 0.25 & 0.40 & 0.20 \end{bmatrix}$$

and suppose we have an external demand of:

$$\bar{d} = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

Then the total amount that must be produced is given by:

$$\bar{p} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.10 & 0.15 & 0.12 \\ 0.20 & 0 & 0.30 \\ 0.25 & 0.40 & 0.20 \end{bmatrix} \right)^{-1} \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 281.30 \\ 464.86 \\ 695.34 \end{bmatrix}$$

So the production of the three Sectors should be set at these values.

Note: It's interesting to note that most of the production is for internal rather than external demand because external demand is only

$$\begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

so that

$$\begin{bmatrix} 281.30 \\ 464.86 \\ 695.34 \end{bmatrix} - \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 181.30 \\ 264.86 \\ 395.34 \end{bmatrix}$$

is being used up internally.

This is because the internal demands are so high - this economy is not very efficient!

Here is an example with smaller internal demands; It's a much more efficient economy:

Example 1.3. Suppose we have three Sectors with consumption matrix:

$$M = \begin{bmatrix} 0.01 & 0.002 & 0.04 \\ 0.02 & 0.004 & 0 \\ 0 & 0.01 & 0.02 \end{bmatrix}$$

and suppose we have an external demand of:

$$\bar{d} = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

Then the total amount that must be produced is given by:

$$\bar{p} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.01 & 0.002 & 0.04 \\ 0.02 & 0.004 & 0 \\ 0 & 0.01 & 0.02 \end{bmatrix} \right)^{-1} \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 113.873 \\ 203.09 \\ 308.195 \end{bmatrix}$$

So the production of the three Sectors should be set at these values. Notice that most of this production goes directly to the external demand.

It's worth noting that in most reasonable economies $(I - M)^{-1}\bar{d}$ is nonnegative (has nonnegative entries) for reasonable $\bar{d}$. However this isn't always the case, and often we can see why fairly easily.

Example 1.4. Consider the consumption matrix:

$$M = \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0 & 0.2 & 0.2 \\ 0.1 & 0.3 & 1 \end{bmatrix}$$

If we do a sample calculation we see:

$$\bar{p} = (I - M)^{-1} \begin{bmatrix} 100 \\ 100 \\ 300 \end{bmatrix} = \begin{bmatrix} -611.11 \\ -796.3 \\ -3685.2 \end{bmatrix}$$

Evidently this doesn't make sense quantitatively, but what is going wrong qualitatively? Consider that Sector 3 requires an entire unit of Product 3 to make a single unit of Product 3. Given that Sectors 1 and 2 also require Product 3, this then totally precludes the possibility of any external demand being filled, or even any internal demand being satisfied, and this is reflected in the calculation.

It might be tempting to believe that we can be on the lookout for values of 1 or more, and that those are problematic, but not all examples are so obvious.

Example 1.5. The reason why this economy has issues is more subtle and is left to the reader:

$$M = \begin{bmatrix} 0.3 & 0.7 & 0.1 \\ 0.8 & 0.2 & 0.2 \\ 0.1 & 0.3 & 0.1 \end{bmatrix}$$

1.2.2 Method - Noninvertible $I - M$

Now let's explore the case where $I - M$ is noninvertible. Again note that this is highly unlikely since most matrices are invertible.

Consider again the equation:

$$(I - M)\bar{p} = \bar{d}$$

This may have no solutions or infinitely many solutions. In what situations might these arise? Recall from basic linear algebra that this is not an easy question. The answer may depend both on the particulars of $I - M$ and $\bar{d}$.

Here is one example.

Example 1.6. Consider the matrix:

$$M = \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{with} \quad I - M = \begin{bmatrix} 0.9 & 0 \\ 0 & 0 \end{bmatrix}$$

It is obvious that $I - M$ is noninvertible. Consider:

$$\bar{d}_1 = \begin{bmatrix} 100 \\ 0 \end{bmatrix} \quad \text{and} \quad \bar{d}_2 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

It is not hard to see (try it!) that $(I - M)\bar{p} = \bar{d}_1$ has infinitely many solutions whereas $(I-M)\bar{p} = \bar{d}_2$ has none. This can been seen from both a computational perspective (try to solve) but also from a quantitative perspective (examine M).

Rather than try to break down every case we'll just focus on one situation, that when $\bar{d} = \bar{0}$.

Observe that $(I - M)\bar{p} = \bar{0}$ has a solution iff $I\bar{p} = M\bar{p}$ which occurs iff that solution $\bar{p}$ is an eigenvector of $\bar{M}$ corresponding to the eigenvalue $\lambda = 1$. In such a case not only will $\bar{p}$ be a solution but any multiple of $\bar{p}$ will, too, since any multiple of an eigenvector is an eigenvector.

This case is of course even more rare since not only would $I - M$ need to be noninvertible but it would have to have an eigenvalue of $\lambda = 1$.

Example 1.7. Consider the consumption matrix

$$M = \begin{bmatrix} 0.1 & 0.4 & 0 \\ 0.2 & 0.4 & 0.9 \\ 0.7 & 0.2 & 0.1 \end{bmatrix}$$

This matrix has an eigenvalue of 1 with corresponding unit (length 1) eigenvector

$$\bar{p} = \begin{bmatrix} 0.3605 \\ 0.8111 \\ 0.4606 \end{bmatrix}$$

The fact that any multiple of this vector is an eigenvector indicates that the three sections can produce in combination any multiple of this.

For example they can set production at any of the following:

$$\begin{bmatrix} 3.605 \\ 8.111 \\ 4.606 \end{bmatrix} \text{ or } \begin{bmatrix} 36.05 \\ 81.11 \\ 46.06 \end{bmatrix} \text{ or } \begin{bmatrix} 360.5 \\ 811.1 \\ 460.6 \end{bmatrix} \text{ or } \begin{bmatrix} 3605.0 \\ 8111.0 \\ 4606.0 \end{bmatrix} \text{ or } \begin{bmatrix} 7.21 \\ 16.222 \\ 9.212 \end{bmatrix}$$

This makes sense because the economy can simply scale up everything simultaneously.

Of course, importantly, this is self-contained, there is no external demand being handled!

1.3 Notes About $(I - M)^{-1}$

1.3.1 Interpretation of Entries

Observe that the equation

$$\bar{p} = (I - M)^{-1}\bar{d}$$

gives the relationship between the external demand and the amount that the Sectors must produce.

However the entries in $(I - M)^{-1}$ itself have their own interpretation. To understand what they mean, suppose that we have some M and some $\bar{d}$. For the sake of simplicity assume M is 3×3.

We know that production must be set at

$$\bar{p} = (I - M)^{-1}\bar{d}$$

Let's investigate what happens if the external demand for Product 1 changes by $+1$.

The new external demand is

$$\bar{d} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

How does $\bar{p}$ change? Well

$$\begin{aligned} \bar{p}_{new} &= (I - M)^{-1}\left(\bar{d} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \\ &= (I - M)^{-1}\bar{d} + (I - M)^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \bar{p} + (I - M)^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Note that the vector

$$(I - M)^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

contains the first column of $(I-M)^{-1}$ which shows us that the first column in $(I-M)^{-1}$ indicates how the production must change in each of the three Sectors if the external demand in the first Sector changes by $+1$.

Consider this example:

Example 1.8. Suppose

$$M = \begin{bmatrix} 0.1 & 0.15 & 0.12 \\ 0.2 & 0 & 0.3 \\ 0.25 & 0.4 & 0.2 \end{bmatrix}$$

If we calculate we find:

$$(I-M)^{-1} = \begin{bmatrix} 1.2660 & 0.3128 & 0.3072 \\ 0.4375 & 1.2850 & 0.5473 \\ 0.6144 & 0.7400 & 1.6200 \end{bmatrix}$$

So now if the external demand for Product 1 changes by $+1$ the production in the three Sectors must change by $+1.266$, $+0.4375$ and $+0.6144$ respectively.

This argument extends to the following:

Fact 1.3.1.1. If the external demand for Sector j changes by $+1$ then column j of $(I-M)^{-1}$ indicates how production must change in all Sectors in order to compensate.

Or entry-by-entry:

Fact 1.3.1.2. If the external demand for Sector j changes by $+1$ then the i^{th} entry of column j of $(I-M)^{-1}$ indicates how the production in Sector i must change in order to compensate.

In addition it's easy to see:

Fact 1.3.1.3. This change is linear in nature in that if the external demand for Sector j changes by, for example, $+2$ then we can simply double column j.

Example 1.9. If an economy with five Sectors has consumption matrix

$$M = \begin{bmatrix} 0.0200 & 0 & 0.1000 & 0.0200 & 0 \\ 0.0600 & 0 & 0.0400 & 0 & 0.0700 \\ 0 & 0.0150 & 0 & 0.0110 & 0 \\ 0 & 0.0220 & 0 & 0 & 0.0800 \\ 0.0300 & 0 & 0.0320 & 0.0100 & 0 \end{bmatrix}$$

then

$$(I - M)^{-1} = \begin{bmatrix} 1.0206 & 0.0020 & 0.1022 & 0.0216 & 0.0019 \\ 0.0634 & 1.0008 & 0.0486 & 0.0025 & 0.0703 \\ 0.0010 & 0.0153 & 1.0008 & 0.0110 & 0.0020 \\ 0.0039 & 0.0221 & 0.0039 & 1.0009 & 0.0816 \\ 0.0307 & 0.0008 & 0.0351 & 0.0110 & 1.0009 \end{bmatrix}$$

so for example if the external demand for the fourth Sector changes by $+1$ then the production of the five Sectors will need to change by

$$\text{Column 4} = \begin{bmatrix} 0.0216 \\ 0.0025 \\ 0.0110 \\ 1.0009 \\ 0.0110 \end{bmatrix}$$

respectively. For example Sector 1 must change production by $+0.02155$, Sector 2 must change production by $+0.002506$, and so on.

Notice that the production of Sector 4 must go up by more than 1. This makes sense since it must produce enough to cover the new external demand for Product 4 plus enough to cover the internal demands of all the Sectors (including itself) as they work together in harmony to increase production.

And for example if the external demand for the Sector 4 product changes by -2 then the production of the five Sectors will need to change by

$$-2(\text{Column 4}) = -2 \begin{bmatrix} 0.0216 \\ 0.0025 \\ 0.0110 \\ 1.0009 \\ 0.0110 \end{bmatrix} = \begin{bmatrix} -0.0431 \\ -0.0050 \\ -0.0221 \\ -2.0019 \\ -0.0220 \end{bmatrix}$$

respectively.

1.3.2 Calculation of Entries of an Inverse

Calculating a matrix inverse can be fairly intensive so it's useful to remember a sneaky formula for calculating entries one by one. This can be useful when coupled with the above meaning of the entries in $(I - M)^{-1}$.

Definition 1.3.2.1. If A is an $n \times n$ matrix then the (i, j)-*cofactor* of A is

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

where A_{ij} is the matrix A with row i and column j removed.

Theorem 1.3.2.1. Let A be an invertible $n \times n$ matrix, then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

where adj(A) is the *adjugate* of A and is defined as follows. Note the peculiar subscript order.

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & ... & C_{n1} \\ C_{12} & C_{22} & ... & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & ... & C_{nn} \end{bmatrix}$$

Proof. Omitted. □

In a brief and compact form the adjugate method for the matrix inverse states that the (i, j) entry of A^{-1} may be found by:

$$\begin{aligned} \left(A^{-1}\right)_{ij} &= \frac{1}{\det(A)} C_{ji} \\ &= \frac{1}{\det(A)} (-1)^{i+j} \det(A_{ji}) \\ &= (-1)^{i+j} \frac{\det(A_{ji})}{\det(A)} \end{aligned}$$

where A_{ji} is the matrix A with row j and column i removed.

Or in a more expanded form:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} +\det(A_{11}) & -\det(A_{21}) & +\det(A_{31}) & \dots \\ -\det(A_{12}) & +\det(A_{22}) & -\det(A_{32}) & \dots \\ +\det(A_{13}) & -\det(A_{23}) & +\det(A_{33}) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example 1.10. Suppose the consumption matrix for three Sectors is:

$$M = \begin{bmatrix} 0.10 & 0.15 & 0.12 \\ 0.20 & 0 & 0.30 \\ 0.25 & 0.40 & 0.20 \end{bmatrix}$$

Suppose the external demand for Sector 3 change by +1. How must production in Sector 2 change?

This value is stored in row 2 of column 3 of $(I - M)^{-1}$. That is, the $(2, 3)$-entry of $(I - M)^{-1}$. To find this value first note that

$$I - M = \begin{bmatrix} 0.90 & -0.15 & -0.12 \\ -0.20 & 1.00 & -0.30 \\ -0.25 & -0.40 & 0.80 \end{bmatrix}$$

For this matrix (call it A) we need $(A^{-1})_{23}$ and we know

$$(A^{-1})_{23} = (-1)^{3+2} \frac{\det(A_{32})}{\det(A)}$$

Well

$$\det(A_{32}) = \det \begin{bmatrix} 0.90 & -0.12 \\ -0.20 & -0.30 \end{bmatrix} = -0.2940$$

and

$$\begin{aligned} \det(A) = &+ (0.9)\,[(1)(0.8) - (-0.3)(-0.4)] \\ &- (-0.15)\,[(-0.2)(0.8) - (-0.3)(-0.25)] \\ &+ (-0.12)\,[(-0.2)(-0.4) - (1)(-0.25)] \\ &= 0.5372 \end{aligned}$$

So then our answer is

$$(A^{-1})_{23} = (-1)^{2+3} \frac{\det(A_{32})}{\det(A)} = -\frac{-0.2940}{0.5372} = 0.5473$$

meaning that if the external demand for Sector 3 changes by +1 then the production in Sector 2 must change by +0.5473. Other Sectors must change production too, of course, but this lets us know about just this Sector.

1.3.3 Expressing as an Infinite Sum

There is an interesting fact about certain matrix inverses which is worth mentioning here because it leads to an interesting observation.

We know from basic Taylor series that

$$\tfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \qquad \text{for } |x| < 1$$

which can also be written as

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots \qquad \text{for } |x| < 1$$

What this is saying is that $1 + x + x^2 + x^3 + \ldots$ is the multiplicative inverse of $1 - x$ when $|x| < 1$.

There is a similar fact about matrices:

Fact 1.3.3.1. Under certain conditions for a square matrix M we have:

$$(I - M)^{-1} = I + M + M^2 + M^3 + \ldots$$

Fact 1.3.3.2. Observe that the above statement can be rewritten:

$$\begin{aligned}
(I - M)(I + M + M^2 + M^3 + \ldots) &= I \\
(I - M)\left[\sum_{k=0}^{\infty} M^k\right] &= I \\
\lim_{n\to\infty} (I - M)\left[\sum_{k=0}^{n} M^k\right] &= I \\
\lim_{n\to\infty} \sum_{k=0}^{n} M^k - M^{k+1} &= I \\
\lim_{n\to\infty} \left(M^0 - M^1 + M^1 - M^2 + \ldots + M^n - M^{n+1}\right) &= I \\
\lim_{n\to\infty} I - M^{n+1} &= I \\
I - \lim_{n\to\infty} M^{n+1} &= I \\
\lim_{n\to\infty} M^{n+1} &= 0
\end{aligned}$$

We can therefore conclude that $(I - M)^{-1} = I + M + M^2 + M^3 + \ldots$ iff $\lim_{n\to\infty} M^{n+1} = 0$. It turns out (this is not obvious) that this occurs iff the maximum absolute value of all the eigenvalues of M is less than 1.

This will be true for all realistic consumption matrices.

If the values in M are small and especially if the matrix is very sparse, meaning if it has lots of zeros, then the sum converges very quickly and finding the sum up until a certain point can be an efficient way to approximate the inverse.

Example 1.11. For example if

$$\begin{bmatrix} 0.100 & 0 & 0.050 & 0 \\ 0.010 & 0 & 0.020 & 0.025 \\ 0 & 0.080 & 0 & 0 \\ 0.030 & 0 & 0 & 0.020 \end{bmatrix}$$

Then if we calculate $I + M + M^2 + ...$ we find that $I + M + M^2 + ... + M^4$ $I + M + M^2 + ... + M^5$ match to four decimal places:

$$I + M + M^2 + M^3 + M^4 + M^5 = \begin{bmatrix} 1.1112 & 0.0045 & 0.0556 & 0.0001 \\ 0.0120 & 1.0017 & 0.0206 & 0.0256 \\ 0.0010 & 0.0801 & 1.0017 & 0.0020 \\ 0.0340 & 0.0001 & 0.0017 & 1.0204 \end{bmatrix}$$

So we can reasonably assume that the sum has "settled" and that this would be a good approximation for $(I - M)^{-1}$, and in fact this matrix actually agrees with $(I - M)^{-1}$ to every digit shown.

So now if we have a very sparse matrix M as above then if we're given some $\bar{d}$ we can find the corresponding $\bar{p}$ approximately and quickly.

Example 1.12. If M is as above and

$$\bar{d} = \begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \end{bmatrix}$$

Then the corresponding $\bar{p}$ can be approximated:

$$\bar{p} = (I - M)^{-1}\bar{d}$$
$$\bar{p} \approx (I + M + M^2 + M^3 + M^4 + M^5)\bar{d}$$
$$\bar{p} \approx \begin{bmatrix} 1.1112 & 0.0045 & 0.0556 & 0.0001 \\ 0.0120 & 1.0017 & 0.0206 & 0.0256 \\ 0.0010 & 0.0801 & 1.0017 & 0.0020 \\ 0.0340 & 0.0001 & 0.0017 & 1.0204 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \end{bmatrix}$$
$$\bar{p} \approx \begin{bmatrix} 12.8741 \\ 21.7938 \\ 31.7434 \\ 41.2103 \end{bmatrix}$$

1.3.4 Meaning of the Infinite Sum

What this infinite sum means is that in our original solution:

$$\bar{p} = (I - M)^{-1}\bar{d}$$

We could have rewritten this as:

$$\bar{p} = (I + M + M^2 + M^3 + ...)\bar{d}$$

Why is this interesting? Consider the original question from another perspective. We wish to produce external demand $\bar{d}$. To do so would suggest perhaps $\bar{p} = \bar{d}$. However this does not take into account the fact that we need to produce not just $\bar{d}$ but also enough to feed the internal demand, so perhaps $\bar{p} = \bar{d} + M\bar{d}$. But we also need to feed the internal demand to feed that internal demand, this is $M(M\bar{d}) = M^2\bar{d}$, and so on, so really:

$$\bar{p} = \bar{d} + M\bar{d} + M^2\bar{d} + M^3\bar{d} + ...$$

Which is exactly the same as the equation above.

Note that we could even have come at this a different way. Since we know that $\bar{p} = M\bar{p} + \bar{d}$ we can recursively plug $\bar{p}$ into the formula to achieve the same result:

$$
\begin{aligned}
\bar{p} &= M\bar{p} + \bar{d} \\
&= M(M\bar{p} + \bar{d}) + \bar{d} \\
&= M^2\bar{p} + M\bar{d} + \bar{d} \\
&= M^2(M\bar{p} + \bar{d}) + M\bar{d} + \bar{d} \\
&= M^3\bar{p} + M^2\bar{d} + M\bar{d} + \bar{d} \\
&= ... \\
&= (I + M + M^2 + M^3 + ...)\bar{d}
\end{aligned}
$$

1.4 Matlab

To enter a matrix in Matlab we can use semicolons to separate rows and either spaces or commas to separate columns. Alternately we can use newlines to separate rows.

The following all assign the same matrix:

```
>> A = [1 0 -2;-2 5 0];

>> A = [1,0,-2;-2,5,0];

>> A = [
1 0 -2
-2 5 0];
```

The inverse of a matrix can be found as follows:

```
>> A = [1 0 -2;-2 5 0;1 2 3];
>> inv(A)
ans =
    0.4545   -0.1212    0.3030
    0.1818    0.1515    0.1212
   -0.2727   -0.0606    0.1515
```

The identity matrix doesn't need to be typed in all the way:

```
>> eye(3)
ans =
     1     0     0
     0     1     0
     0     0     1
```

We can then do something like this:

```
>> M = [1 0 -2;-2 5 0;1 2 3];
>> inv(eye(3)-M)
ans =
   -0.5000    0.2500   -0.5000
   -0.2500   -0.1250   -0.2500
    0.5000         0         0
```

We can then do a calculation such as this from an earlier example:

```
>> M = [
0.10 0.15 0.12
0.20 0 0.30
0.25 0.40 0.20];
>> d = [100;200;300];
>> inv(eye(3)-M)*d
ans =
  281.2995
  464.8608
  695.3365
```

We can test long sums of powers of matrices easily, like this earlier example:

```
>> M = [
0.1 0 0.05 0
0.01 0 0.02 0.025
0 0.08 0 0
0.03 0 0 0.02];
>> eye(4)+M+M^2+M^3+M^4
ans =
    1.1112    0.0044    0.0556    0.0001
    0.0120    1.0016    0.0206    0.0256
    0.0010    0.0801    1.0016    0.0020
    0.0340    0.0001    0.0017    1.0204
```

Eigenvalues and eigenvectors can be found together but we have to know how to interpret the result. In the following the matrix `d` contains one eigenvalue per entry on the diagonal and the matrix `p` contains the eigenvectors where the first column contains an eigenvector corresponding to the first entry in `d`, and so on:

```
>> M = [
0.1 0.4 0
0.2 0.4 0.9
0.7 0.2 0.1];
>> [p,d] = eig(M)
p =
  -0.3605 + 0.0000i  -0.2835 - 0.4119i  -0.2835 + 0.4119i
  -0.8111 + 0.0000i   0.6614 + 0.0000i   0.6614 + 0.0000i
  -0.4606 + 0.0000i  -0.3780 + 0.4119i  -0.3780 - 0.4119i
d =
   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.2000 + 0.4359i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.2000 - 0.4359i
```

The determinant is easy:

```
>> A = [1 0 -2;-2 5 0;1 2 3];
>> det(A)
ans =
    33
```

1.5 Exercises

Exercise 1.1. Which of the following consumption matrices could yield a nonzero production vector for a closed economy? Justify.

$$M_1 = \begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.5 & 0.7 & 0.3 \\ 0.3 & 0.2 & 0.4 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0.04 & 0.02 & 0.06 \\ 0.10 & 0.14 & 0.06 \\ 0.06 & 0.04 & 0.08 \end{bmatrix} \quad M_3 = \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.0 \end{bmatrix}$$

Exercise 1.2. The following consumption matrix can yield a nonzero production vector for a closed economy. Find two different interesting corresponding nonzero production vectors.

$$M = \begin{bmatrix} 0.3157 & 0.947 \\ 0.6314 & 0.1263 \end{bmatrix}$$

Note: Due to approximations in technology you might not get an eigenvalue of exactly 1 but the intention is that 1 is an eigenvalue.

Exercise 1.3. The following consumption matrix can yield a nonzero production vector for a closed economy. Find two different interesting corresponding nonzero production vectors.

$$M = \begin{bmatrix} 0.5956 & 0.2978 & 0.1787 \\ 0.2978 & 0 & 0.4169 \\ 0.8934 & 0.1191 & 0.05956 \end{bmatrix}$$

Note: Due to approximations in technology you might not get an eigenvalue of exactly 1 but the intention is that 1 is an eigenvalue.

Exercise 1.4. Suppose an economy has two Sectors producing products Product 1 and Product 2 respectively.

- It takes 0.10 units of Product 1 and 0.05 units of Product 2 to produce 1 unit of Product 1.
- It takes 0.06 units of Product 1 and 0.12 units of Product 2 to produce 1 units of Product 2.

(a) What should the total Production be set at in order to satisfy an external demand of 20 units of Product 1 and 30 units of Product 2?

(b) Find $(I - M)^{-1}$ by hand and interpret each entry in the first column.

Exercise 1.5. Suppose an economy has two Sectors producing Products Product 1 and Product 2 respectively.

- It takes 0.22 units of Product 1 and 0.15 units of Product 2 to produce 1 unit of Product 1.
- It takes 0.16 units of Product 1 and 0.26 units of Product 2 to produce 1 units of Product 2.

(a) What should the total Production be set at in order to satisfy an external demand of 120 units of Product 1 and 150 units of Product 2?

(b) Would you say this economy is efficient or not and why?

(c) Suppose the Sectors scale down their input requirements to 10% of what they were. Repeat (a) and (b).

(d) Find $(I - M)^{-1}$ by hand and interpret each entry in the first row.

Exercise 1.6. Suppose the consumption matrix for a certain economy is

$$M = \begin{bmatrix} 0.02 & 0 & 0.05 \\ 0.1 & 0.08 & 0 \\ 0 & 0.12 & 0.04 \end{bmatrix}$$

Using the adjugate method - if the demand for Sector 2 changes by -2 how must the production of Sector 3 respond?

Exercise 1.7. Suppose an economy has three Sectors producing products Product 1, Product 2 and Product 3 respectively.

- It takes 0.20 units of Product 1, 0.15 units of Product 2, and 0.10 units of Product 3 to produce 1 unit of Product 1.
- It takes 0.10 units of Product 1, 0.05 units of Product 2, and 0.12 units of Product 3 to produce 1 unit of Product 2.
- It takes 0.14 units of Product 1 and 0.08 units of Product 2 to produce 1 unit of Product 3.

(a) What should the total Production be set at in order to satisfy an external demand of 100 units of Product 1, 120 units of Product 2 and 150 units of Product 3?

(b) Using the adjugate method calculate how production in Sector 1 must respond if the demand for Sector 3 changes by $+1$. How about by $+2$? How about by -1?

(c) Find $(I - M)^{-1}$ using technology.

Exercise 1.8. Suppose an economy has three Sectors producing products Product 1, Product 2 and Product 3 respectively.

- It takes 0.02 units of Product 1, 0.06 units of Product 2, and 0.10 units of Product 3 to produce 1 unit of Product 1.
- It takes 0.40 units of Product 2 and 0.04 units of Product 3 to produce 1 unit of Product 2.
- It takes 0.18 units of Product 1, 0.01 units of Product 2, and 0.10 units of Product 3 to produce 1 unit of Product 3.

(a) What should the total production be set at in order to satisfy an external demand of 200 units of Product 1, 180 units of Product 2 and 175 units of Product 3?

(b) Using the adjugate method calculate how production in Sector 3 must change if the demand for Sector 2 changes by +1. How about by +2? How about by −1?

(c) Find $(I - M)^{-1}$ using technology.

Exercise 1.9. Suppose the consumption matrix for two Sectors is given by:

$$M = \begin{bmatrix} 0.10 & 0.06 \\ 0.05 & 0.12 \end{bmatrix}$$

(a) Find the smallest i so that $I + M + ... + M^i$ is the same as $I + M + ... + M^{i-1}$ when rounded to the fourth decimal digit.

(b) Write down the matrix $I + M + ... + M^i$ using your i from (a).

(c) What can you say about $(I - M)^{-1}$?

Exercise 1.10. Suppose the consumption matrix for four Sectors is given by:

$$M = \begin{bmatrix} 0.02 & 0 & 0.02 & 0.04 \\ 0 & 0.01 & 0.05 & 0.04 \\ 0.03 & 0 & 0.01 & 0.02 \\ 0.02 & 0.05 & 0.01 & 0 \end{bmatrix}$$

Using the infinite series method find a quick approximation for how production in Sector 1 must change if the demand for Sector 1 changes by +1.
Note: I'll let you decide how far to take your powers of M, just explain why you made whatever choice you did.

Exercise 1.11. In any sensible economy for each i the (i, i)-entry in $(I - M)^{-1}$ is always greater than 1. Explain why this is, economically speaking.

Exercise 1.12. Suppose for some consumption matrix M you find that $I + M + M^2 + ...$ does not converge. Intuitively speaking what does this say about the production issues associated to this economy? If it helps, give an example and use it to clarify.

Exercise 1.13. Suppose M is a consumption matrix with the property that all entries are between 0 and 1 inclusive except for some i for which $m_{ii} > 1$. Use a mathematical argument to explain why $\lim_{i\to\infty} M^i$ will not converge and hence $I + M + M^2 + \ldots$ will not converge.

Exercise 1.14. Suppose the consumption matrix for an economy with two Sectors is given by

$$M = \begin{bmatrix} 1.02 & 0.06 \\ 0.05 & 0.01 \end{bmatrix}$$

(a) From an economic standpoint why is this M unrealistic?
Hint: Consider what the value of m_{11} means.

(b) Find $(I - M)^{-1}$.

(c) The previous question implies that $I + M + M^2 + \ldots$ does not converge, and yet $(I - M)^{-1}$ exists. Why does this not contradict the equality given in the chapter?

Exercise 1.15. Suppose the consumption matrix for an economy with two Sectors is given by

$$M = \begin{bmatrix} 0.06 & 1.02 \\ 0.05 & 0.10 \end{bmatrix}$$

(a) This matrix is very similar to the matrix in the previous question yet this one gives reasonable results whereas the previous question does not. Explain this difference in economic terms. In other words why is it economically reasonable that $m_{12} > 1$ but not that $m_{11} > 1$?

(b) Find $(I - M)^{-1}$.

(c) Find $\bar{p}$ which corresponds to $\bar{d} = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$.

(d) Show with some calculation that it seems that $I+M+M^2+\ldots$ does converge to $(I - M)^{-1}$ in this case.

Exercise 1.16. Suppose the consumption matrix for an economy with two Sectors is given by the following where all of a, b, c, d are between 0 and 1 inclusive

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(a) Find a formula for $(I - M)^{-1}$.

(b) Explain why this result is reasonable if and only if

$$(1-a)(1-d) > bc$$

That is, if the inequality is not satisfied what can you say about the values in the inverse and why are those values economically nonsensical?

(c) Give some examples of how this makes sense in terms of the economy and the Sectors. This is a tricky question with an interesting and commonsense answer. Think about how a and b are related to one another, what this means in economic terms, and how this feeds into the inequality you find. Similarly for c and d.

Exercise 1.17. Suppose instead of being given the consumption matrix M for an economy we are given the matrix $(I - M)^{-1}$. Call this the *marginal production response matrix.* (I have no idea if it has another name!)

(a) How can you find M?

(b) Apply this method to find M if you're given the marginal production response matrix

$$\begin{bmatrix} 1.06 & 0.02 & 0.04 \\ 0.02 & 1.06 & 0.09 \\ 0.11 & 0.02 & 1.03 \end{bmatrix}$$

Exercise 1.18. Suppose an economy has two Sectors producing Products Product 1 and Product 2 respectively.

- It takes 0.10 units of Product 1 and 0.05 units of Product 2 to produce 1 unit of Product 1.
- It takes 0.06 units of Product 1 and 0.12 units of Product 2 to produce 1 unit of Product 2.

Suppose the total production is fixed at 1000 units of Product 1 and 1500 units of Product 2 of which some (as much as is needed) is used internally and the rest (normally called the external demand) is stored as surplus. How much of each is there as surplus?

Exercise 1.19. Suppose an economy has two Sectors producing products Product 1 and Product 2 respectively.

- It takes 0.10 units of Product 1 and 0.20 units of Product 2 to produce 1 unit of Product 1.
- It takes x units of Product 1 and 0.05 units of Product 2 to produce 1 unit of Product 2.

(a) How large can x be for this to be economically sensible, meaning it can find a solution for any external demand? Justify.

(b) If the 0.20 is replaced by y, what relationship between x and y would be economically sensible?