Markov Chains

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6.1 Introduction

Markov chains have many applications but we'll start with one which is easy to understand.

6.1.1 The Problem

Suppose there are two states (think countries, or US states, or cities, or whatever) 1 and 2 with a total population of 1 distributed as 0.7 in State 1 and 0.3 in State 2.

Suppose that at the end of the year 10% of the people in State 1 move out of State 1 and into State 2 (the rest remain) and 5% of the people in State 2 move out of State 2 and into State 1 (the rest remain).

What will the new population distribution be? We can find this out easily:

State 1 now has $0.7 - 0.10(0.7) + 0.05(0.3) = 0.6450$

State 2 now has $0.3 - 0.05(0.3) + 0.10(0.7) = 0.3550$

There's another way to write this, however. If we rephrase this as 90% of the people in State 1 stay in State 1 and 5% of the people in State 2 move to State 1, and similarly for State 2 then we get:

State 1 now has $0.90(0.7) + 0.05(0.3) = 0.6450$

State 2 now has $0.10(0.7) + 0.95(0.3) = 0.3550$

Now then, suppose this happened again the next year. Again, this is easy to find out:

State 1 now has $0.90(0.6450) + 0.05(0.3550) = 0.5983$

State 2 now has $0.10(0.6450) + 0.95(0.3550) = 0.4017$

Suppose this keeps happening year-by-year for years. This calculation would not only be annoying to repeat but perhaps it's possible to gain some insight without doing it over and over.

6.1.2 The Problem Rephrased with Matrices and Vectors

For starters let's notice that we can convert the calculations very easily to linear algebra. If the populations start with the vector:

$$\begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

Then after one year the new populations are:

$$\begin{bmatrix} 0.90 & 0.05 \\ 0.10 & 0.95 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.90(0.7) + 0.05(0.3) \\ 0.10(0.7) + 0.95(0.3) \end{bmatrix} = \begin{bmatrix} 0.6450 \\ 0.3550 \end{bmatrix}$$

In other words if the population is $\bar{x}_0$ then after one year the population is:

$$\bar{x}_1 = T\bar{x}_0$$

Where

$$T = \begin{bmatrix} 0.90 & 0.05 \\ 0.10 & 0.95 \end{bmatrix}$$

Furthermore after another year the population is:

$$\bar{x}_2 = T\bar{x}_1 = T(T\bar{x}_0) = T^2\bar{x}_0$$

and in general after n years:

$$\bar{x}_n = T^n\bar{x}_0$$

6.1.3 Higher Dimensions

The first nice thing to notice about moving to matrices and vectors is that the problem generalizes easily to higher dimensions.

First let's make one thing clear that it's easy to mess up. In the matrix T (called the transition matrix) column j contains the information about where the population in state j goes to. You could think of as an "exit vector" from state j. More specifically the (i, j) row (i^{th} entry in the jth column) contains the proportion of people who moved from state j to state i in exactly one iteration.

So then when we move to more states we simply add rows and columns as needed for the number of states we have.

Example 6.1. Consider the following diagram which shows the populations shifting between three states:
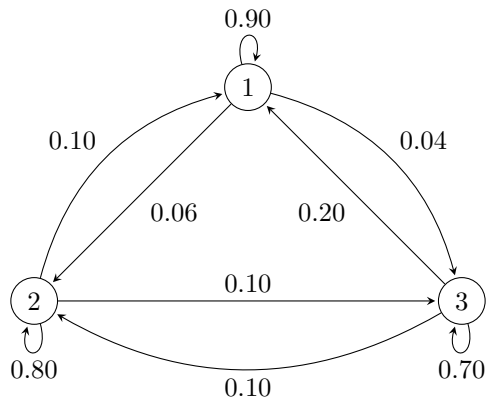
Let's determine the steady state of this situation in terms of how the populations will distribute in the long term.

The transition matrix is:

$$T = \begin{bmatrix} 0.90 & 0.10 & 0.20 \\ 0.06 & 0.80 & 0.10 \\ 0.04 & 0.10 & 0.70 \end{bmatrix}$$

So now for a given initial population distribution we can find out the distribution after one, two and five iterations easily: If

$$\bar{x}_0 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.7 \end{bmatrix}$$

then

$$T\bar{x}_0 \approx \begin{bmatrix} 0.2500 \\ 0.2360 \\ 0.5140 \end{bmatrix}$$

$$T^2\bar{x}_0 \approx \begin{bmatrix} 0.3514 \\ 0.2552 \\ 0.3934 \end{bmatrix}$$

$$T^5\bar{x}_0 \approx \begin{bmatrix} 0.5007 \\ 0.2692 \\ 0.2302 \end{bmatrix}$$

6.1.4   Long Term Behavior Experiment

So what happens in the long term? Let's take a look as n gets large for our original distribution:

$$\bar{x}_0 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

$$T\bar{x}_0 = \begin{bmatrix} 0.6450 \\ 0.3550 \end{bmatrix}$$

$$T^2\bar{x}_0 = \begin{bmatrix} 0.5983 \\ 0.4017 \end{bmatrix}$$

$$T^{10}\bar{x}_0 \approx \begin{bmatrix} 0.4055 \\ 0.5945 \end{bmatrix}$$

$$T^{100}\bar{x}_0 \approx \begin{bmatrix} 0.3333 \\ 0.6667 \end{bmatrix}$$

$$T^{1000}\bar{x}_0 \approx \begin{bmatrix} 0.3333 \\ 0.6667 \end{bmatrix}$$

Oh, that's interesting. It looks like the population settles whereby $1/3$ is in State 1 and $2/3$ is in State 2. Moreover it really suggests two things:

- $\lim\limits_{n\to\infty} T^n\bar{x}_0 = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$ for this particular $\bar{x}_0$.

- $T\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

The second of these is true, and it's easy to check.

As for the first, we might ask a preliminary question: What if we'd started with a very different distribution of people? Let's try with 0.05 and 0.95 in State 1 and State 2 respectively!

$$\bar{x}_0 = \begin{bmatrix} 0.05 \\ 0.95 \end{bmatrix}$$

$$T\bar{x}_0 = \begin{bmatrix} 0.0925 \\ 0.9075 \end{bmatrix}$$

$$T^{10}\bar{x}_0 = \begin{bmatrix} 0.2776 \\ 0.7224 \end{bmatrix}$$

$$T^{100}\bar{x}_0 \approx \begin{bmatrix} 0.3333 \\ 0.6667 \end{bmatrix}$$

$$T^{1000}\bar{x}_0 \approx \begin{bmatrix} 0.3333 \\ 0.6667 \end{bmatrix}$$

Oh wow! Could it be that the population always approaches this same distribution?

6.2 Steady States and Limits

6.2.1 Formal Stuff

It appears (no proof yet!) in our simple example that there is a vector which is fixed by T and to which perhaps all other states approach as $n \to \infty$. However is this always the case?

Let's lay everything out formally:

Definition 6.2.1.1. A *probability vector* is a vector whose entries lie between 0 and 1 (inclusive) and add to 1. An example would be a distribution of the population.

Definition 6.2.1.2. A *transition matrix* (also known as a *stochastic matrix*) or *Markov matrix* is a matrix in which each column is a probability vector. An example would be the matrix representing how the populations shift year-to-year where the (i, j) entry contains the fraction of people who move from state j to state i in one iteration.

Definition 6.2.1.3. A probability vector $\bar{x}$ is a *steady-state vector* for a transition matrix T if $T\bar{x} = \bar{x}$. Notice that a steady-state vector is an eigenvector corresponding to the eigenvalue $\lambda = 1$.

Definition 6.2.1.4. A *regular transition matrix* is a transition matrix T such that there is some integer $k \geq 1$ such that all entries of T^k are nonzero. For the simplest case if all the entries of T itself are nonzero then T is a regular transition matrix.

Theorem 6.2.1.1. If T is a regular transition matrix then it has $\lambda = 1$ as an eigenvalue and there is a unique steady-state eigenvector $\bar{x}_*$. Moreover for any probability vector $\bar{x}_0$, $\lim\limits_{k \to \infty} T^k \bar{x}_0 = \bar{x}_*$.

Proof. This proof is hard and is omitted. Later in the chapter there is a proof for the 2×2 case. □

Corollary 6.2.1.1. If T is a regular transition matrix then $\lim\limits_{k \to \infty} T^k$ exists, where the columns of the result are all identical and all equal $\bar{x}_*$.

Proof. Suppose T is $n \times n$, then we know from the theorem that for each $1 \leq i \leq n$ we have

$$\lim_{k \to \infty} T^k \bar{e}_i = \bar{x}_*$$

so that

$$\lim_{k \to \infty} T^k \begin{bmatrix} \bar{e}_1 & \dots & \bar{e}_n \end{bmatrix} = \begin{bmatrix} \bar{x}_* & \dots & \bar{x}_* \end{bmatrix}$$

$$\lim_{k \to \infty} T^k I = \begin{bmatrix} \bar{x}_* & \dots & \bar{x}_* \end{bmatrix}$$

which gives the result. □

It follows from the corrollary that computationally speaking if we want to approximate the steady state vector for a regular transition matrix T that all we need to do is look at one column from T^k for some very large k.

Fact 6.2.1.1. If T is a transition matrix but is not regular then there is no guarantee that the results of the Theorem will hold! They might, but no guarantee.

Fact 6.2.1.2. A transition matrix which is not regular may have more than one steady state vector and there are no guarantees about limiting behavior.

Fact 6.2.1.3. If you use Matlab or Wolfram Alpha to find this eigenvector be aware that it will almost certainly not give you a probability vector. For example Matlab typically gives a unit vector. However since any multiple of an eigenvector is an eigenvector you can simply divide by the sum of the values to get an eigenvector which is a probability vector.

6.2.2 Full Theory Example

Example 6.2. Consider the example from earlier with transition matrix

$$T = \begin{bmatrix} 0.90 & 0.10 & 0.20 \\ 0.06 & 0.80 & 0.10 \\ 0.04 & 0.10 & 0.70 \end{bmatrix}$$

Notice that T is regular because all entries of T^1 are nonzero.

Now then, the eigenvalues of this matrix are $\lambda_1 = 1$, $\lambda_2 \approx 0.763246$, $\lambda_3 \approx 0.636754$.

Note that in reality we don't need these - we know (from the theorem) that $\lambda = 1$ is an eigenvalue and so all we really need is the associated steady-state vector,

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meaning an eigenvector which is a probability vector, meaning the entries add to 1.

Matlab tells us an eigenvector for $\lambda_1 = 1$ is approximately:

$$\begin{bmatrix} 0.886659 \\ 0.39013 \\ 0.248264 \end{bmatrix}$$

This is not a probability vector, in fact this is a vector whose magnitude is 1 but the sum of the entries is not. So to make it a probability vector we divide through by the sum (because any nonzero multiple of an eigenvector is also an eigenvector) to get:

$$\begin{bmatrix} 0.581397 \\ 0.255815 \\ 0.162791 \end{bmatrix}$$

Which means that in the long term:

- 58.1397% of the population will end up in 1.

- 25.5815% of the population will end up in 2.

- 16.2791% of the population will end up in 3.

As per the corollary we could also have found these approximately by taking a sufficiently large value of T and looking at a column:

$$T^{1000} \approx \begin{bmatrix} 0.581395 & 0.581395 & 0.581395 \\ 0.255814 & 0.255814 & 0.255814 \\ 0.162791 & 0.162791 & 0.162791 \end{bmatrix}$$

6.3 Transition Matrices and Regularity

The definition of T being regular is that there is some power T^k for which none of the entries are zero. What does this actually mean, though, and what does it mean for T to be non-regular?

We can answer this by investigating the meaning of the entries in T^2, T^3, etc.

Let's look at T^2. If we write:

$$T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} \end{bmatrix}$$

First we know that the (i, j) entry in T represents the proportion of people who move from state j to state i in one interation.

The (i, j)-entry in T^2 equals:

$$t_{i1}t_{1j} + t_{i2}t_{2j} + ... + t_{in}t_{nj}$$

But what does this value represent?

Well notice that:

- $t_{i1}t_{1j}$ represents the proportion of people who move from state j to state 1 and then from state 1 to state i.

- $t_{i2}t_{2i}$ represents the proportion of people who move from state j to state 2 and then from state 2 to state i.

- ...until...

- $t_{in}t_{nj}$ represents the proportion of people who move from state j to state n and then from state n to state i.

It follows that the (i, j)-entry in T^2 represents the proportion of people who move from state j to state i in exactly two iterations.

Similarly the (i, j)-entry in T^3 represents the proportion of people who move from state j to state i in exactly three iterations and in general the (i, j)-entry in T^k represents the proportion of people who move from state j to state i in exactly k iterations.

What this means is that for a transition matrix to be regular that there is some iterative step (the k value) for which it can be said that in exactly k iterations some people move from every state to every other state. Note that this is not the same as saying that the population reaches every state from each state eventually. For example if $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then everyone alternates states but the matrix is not regular.

However we can certainly say that if some states are not reachable from others (ever) then the transition matrix will definitely not be regular since if state i is not reachable from j (ever) then the (i, j) entry of T^k will be zero for all k.

Here some examples to flesh out these ideas:

Example 6.3. Consider:
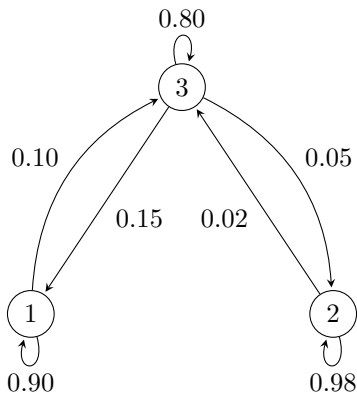
The transition matrix for this is:

$$T = \begin{bmatrix} 0.9 & 0 & 0.15 \\ 0 & 0.98 & 0.05 \\ 0.1 & 0.02 & 0.8 \end{bmatrix}$$

Observe the following two:

$$T^2 = \begin{bmatrix} 0.825 & 0.003 & 0.255 \\ 0.005 & 0.9614 & 0.089 \\ 0.17 & 0.0356 & 0.656 \end{bmatrix} \text{ and } T^5 = \begin{bmatrix} 0.6894 & 0.02171 & 0.4117 \\ 0.03618 & 0.9118 & 0.1662 \\ 0.2744 & 0.06646 & 0.4222 \end{bmatrix}$$

Some things we can observe:

- The matrix is regular because T^2 has no zeros.

- The $(3, 2)$ entry in T^2 is 0.0356 and indicates that in exactly two iterations 0.0356 of the population of state 2 will have moved to state 3.

- The $(3, 1)$ entry in T^5 is 0.2744 and indicates that in exactly five iterations 0.2744 of the population of state 1 will have moved to state 3.

- The $(2, 1)$ entry in T is 0 since we can't get from state 1 to state 2 in exactly one iteration but the $(2, 1)$ entry in T^2 is 0.005 since we can do it in exactly two iterations.

Example 6.4. Consider:
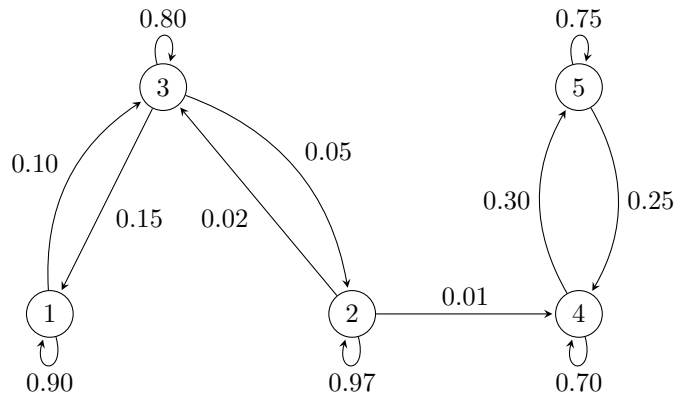
Some things we can observe:

- This matrix is not regular because it is impossible to ever get from states 4,5 to states 1,2,3. Consequently the (i,j) entries of T^k will be zero for $i = 1, 2, 3$ and $j = 4, 5$ for all k.

- It's certainly not possible to get from state 1 to state 5 in one iteration but it is possible in four. Consequently the $(5,1)$ entry in T would be zero but the $(5,1)$ entry in T^4 would not be. It would in fact be $(0.10)(0.05)(0.01)(0.30) = 0.000015$.

- To directly calculate the $(3,1)$ entry in T^3 we look at the paths from state 1 to state 3 which are three iterations long. There are five:

$$1 \rightarrow 3 \rightarrow 1 \rightarrow 3$$
$$1 \rightarrow 3 \rightarrow 2 \rightarrow 3$$
$$1 \rightarrow 3 \rightarrow 3 \rightarrow 3$$
$$1 \rightarrow 1 \rightarrow 1 \rightarrow 3$$
$$1 \rightarrow 1 \rightarrow 3 \rightarrow 3$$

Consequently the $(3,1)$ entry in T^3 equals:

$$(0.10)(0.15)(0.10)$$
$$+(0.10)(0.05)(0.02)$$
$$+(0.10)(0.80)(0.80)$$
$$+(0.90)(0.90)(0.10)$$
$$+(0.90)(0.10)(0.80)$$
$$= 0.2186$$

Interestingly if we write down the transition matrix for this:

$$T = \begin{bmatrix} 0.9 & 0 & 0.15 & 0 & 0 \\ 0 & 0.97 & 0.05 & 0 & 0 \\ 0.1 & 0.02 & 0.8 & 0 & 0 \\ 0 & 0.01 & 0 & 0.7 & 0.25 \\ 0 & 0 & 0 & 0.3 & 0.75 \end{bmatrix}$$

and if we find $T^k \bar{x}_0$ for any $\bar{x}_0$ and for some very large k (emulating a limit):

$$T^k \bar{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.455 \\ 0.545 \end{bmatrix}$$

So this shows (experimentally) that there is a steady-state vector to which all other states converge.

What is happening here is that in the long term everything moves out of states 1,2,3 and into 4,5 which act like their own little Markov chain and $\begin{bmatrix} 0.455 \\ 0.545 \end{bmatrix}$ is the steady state for this chain.

Example 6.5. Consider the simple example with 1 and 2 where, each year, 100% of the population moves to the other state.

The transition matrix here is:

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which has $T^2 = I$, $T^3 = T$, etc. and is hence not regular.

Moreover any initial population will flip year by year.

If the initial population distribution is $\begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$ then the following year it will be $\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$ and then $\begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$ and so on, and this will happen (this flip) for any distribution it starts with. Basically this distribution never settles unless it begins (and stays with) $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

So there is a single steady-state vector, specifically $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, but no other state will converge to it.

6.4 Steady State Proof for The Two-Dimensional Case

Theorem 6.4.0.1. If T is a 2×2 regular transition matrix then T has a steady-state vector $\bar{x}_*$ and moreover for any vector $\bar{x}_0$ we have $\lim\limits_{n \to \infty} T^n \bar{x}_0 = \bar{x}_*$.

Proof. A 2×2 matrix whose columns add to 1 has the form

$$T = \begin{bmatrix} a & 1-b \\ 1-a & b \end{bmatrix}$$

where $0 \leq a \leq 1$ and $0 \leq b \leq 1$.

Notice first that if $a = b = 1$ then

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and if $a = b = 0$ then

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and neither of these are regular so we can safely ignore these cases.

Calculation shows that the eigenvalue-eigenvector pairs of T are:

$$\left\{ 1, \begin{bmatrix} b-1 \\ a-1 \end{bmatrix} \right\} \text{ and } \left\{ a+b-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

Notice that $\lambda = 1$ is an eigenvalue with eigenvector

$$\begin{bmatrix} b-1 \\ a-1 \end{bmatrix}$$

If we divide through by the sum $(a-1) + (b-1)$ (which is not zero because we've excluded $a = b = 1$) we get the probability eigenvector:

$$\begin{bmatrix} \frac{b-1}{(a-1)+(b-1)} \\ \frac{a-1}{(a-1)+(b-1)} \end{bmatrix}$$

We can see here that the sum is now 1 and each value is between 0 and 1, so we have a steady-state vector which we'll denote $\bar{x}_*$.

Next we need to show that for any probability vector $\bar{x}_0$ we have the long-term behavior $\lim\limits_{n\to\infty} T^n \bar{x}_0 = \bar{x}_*$.

First, it follows from our eigenpairs that the diagonalization of T is:

$$T = \begin{bmatrix} b-1 & -1 \\ a-1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a+b-1 \end{bmatrix} \begin{bmatrix} b-1 & -1 \\ a-1 & 1 \end{bmatrix}^{-1}$$

and therefore:

$$
\begin{aligned}
\lim_{n\to\infty} T^n &= \lim_{n\to\infty} \begin{bmatrix} b-1 & -1 \\ a-1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a+b-1 \end{bmatrix}^n \begin{bmatrix} b-1 & -1 \\ a-1 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} b-1 & -1 \\ a-1 & 1 \end{bmatrix} \lim_{n\to\infty} \begin{bmatrix} 1 & 0 \\ 0 & (a+b-1)^n \end{bmatrix} \begin{bmatrix} b-1 & -1 \\ a-1 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} b-1 & -1 \\ a-1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b-1 & -1 \\ a-1 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \frac{b-1}{a+b-2} & \frac{b-1}{a+b-2} \\ \frac{a-1}{a+b-2} & \frac{a-1}{a+b-2} \end{bmatrix}
\end{aligned}
$$

Notice that

$$\lim_{n\to\infty} (a+b-1)^n = 0$$

because $-1 < a+b-1 < 1$ (because we've excluded $a = b = 0$ and $a = b = 1$).

Now then if $\bar{x}_0$ is any probability vector then if we write:

$$\bar{x}_0 = \begin{bmatrix} c \\ 1-c \end{bmatrix}$$

with $0 \le c \le 1$ then calculation shows that:

$$\lim_{n\to\infty} T^n \bar{x}_0 = \begin{bmatrix} \frac{b-1}{a+b-2} & \frac{b-1}{a+b-2} \\ \frac{a-1}{a+b-2} & \frac{a-1}{a+b-2} \end{bmatrix} \begin{bmatrix} c \\ 1-c \end{bmatrix} = \begin{bmatrix} \frac{b-1}{a+b-2} \\ \frac{a-1}{a+b-2} \end{bmatrix} = \bar{x}_*$$

as desired.

$\square$

6.5   Matlab

We addressed earlier how we can find the eigenvalues and eigenvectors of a matrix but it's worth revisiting, especially since we need to get a vector which sums to 1.

Here's an example from earlier.

```
>> T = [
0.90 0.10 0.20
0.06 0.80 0.10
0.04 0.10 0.70];
```

We can find all the eigenpairs as before:

```
>> [p,d] = eig(T)
p =
    0.8867     0.8071    -0.4960
    0.3901    -0.5105    -0.3137
    0.2483    -0.2966     0.8097
d =
    1.0000          0          0
         0     0.7632          0
         0          0     0.6368
```

What we'd like to do is take the first column of `p`, which is the eigenvector corresponding to $\lambda = 1$, the first entry in the diagonal matrix `d`, and divide by its magnitude. Here is how we can do it in one go:

```
>> v = p(:,1)/sum(p(:,1))
v =
    0.5814
    0.2558
    0.1628
```

This notation might be familiar. Writing `p(:,1)` takes every row in the first column of `p`. So what we're doing is taking that vector and dividing it by its own sum.

6.6   Exercises

Exercise 6.1. Consider the following population movement diagram.
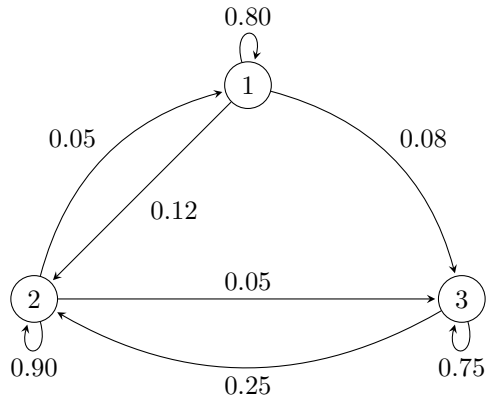
(a) Write down the corresponding transition matrix T.

(b) What does the fact that T is not symmetric say about the population movements?

(c) If the population distribution starts at $\bar{x}_0 = \begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix}$ what will it be after one iteration? How about after two iterations?

(d) Calculate the steady state distribution $\bar{x}^*$.

(e) Using software (your choice) find the smallest value of k such that $T^k \bar{x}_0$ agrees with $\bar{x}^*$ to four decimal places.

Exercise 6.2. Given the transition matrix:

$$T = \begin{bmatrix} 0.9 & 0 & 0.05 & 0 & 0.15 & 0.05 \\ 0 & 0.8 & 0.05 & 0.2 & 0.05 & 0.05 \\ 0.01 & 0.1 & 0.5 & 0 & 0 & 0.05 \\ 0.02 & 0.1 & 0.1 & 0.75 & 0 & 0.05 \\ 0.07 & 0 & 0.1 & 0 & 0.7 & 0.05 \\ 0 & 0 & 0.2 & 0.05 & 0.1 & 0.75 \end{bmatrix}$$

(a) Draw the corresponding population movement diagram.

(b) Is this matrix regular? Justify.

Exercise 6.3. Draw a population movement diagram whose transition matrix is 3×3, regular and symmetric. Also give the transition matrix.

Exercise 6.4. Given the following population movement diagram:
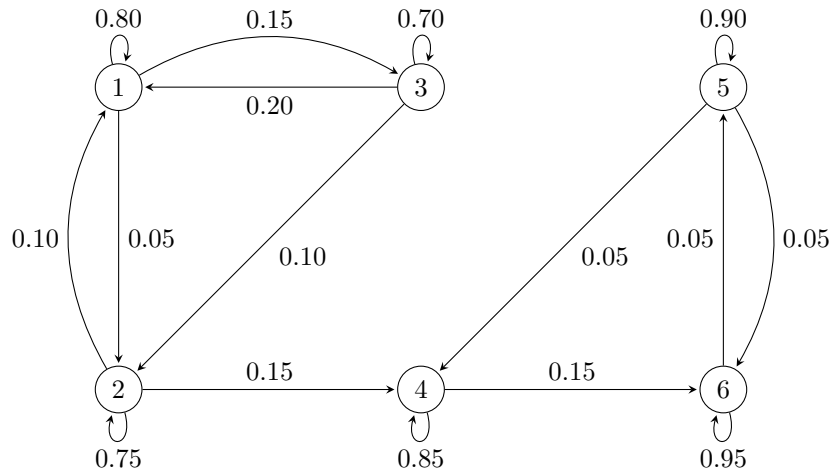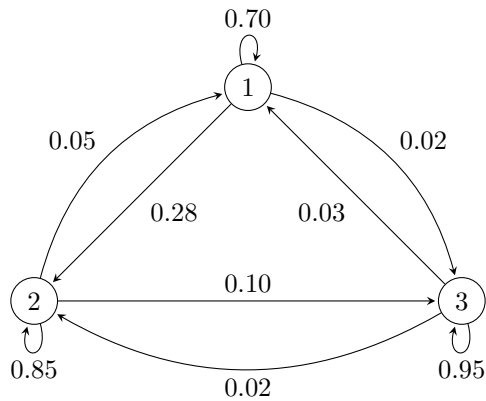
(a) Write down the transition matrix T for this.

(b) Find each of the following without actually taking powers of T:

    (a) The $(3, 2)$ entry of T.

    (b) The $(3, 2)$ entry of T^2.

    (c) The $(1, 3)$ entry of T^2. Be careful!

    (d) The smallest k such that the $(5, 3)$ entry of T^k is nonzeo and what that value is.

    (e) All (i, j) such that the (i, j) entry of T^k is zero for all k.

    (f) Explain intuitively what will happen in the long term to any initial population distribution. Justify intuitively. This question can be answered to various degrees of detail, the most basic being - in which area(s) do the populations tend to eventually move and why?

Exercise 6.5. Find a population movement diagram and the corresponding transition matrix T such that $T^7 = I$ but $T^k \neq I$ for $k < 7$.

Exercise 6.6. We know that a regular transition matrix has a steady state to which all states converge and that a nonregular transition matrix does not necessarily have to. Give an example of a population movement diagram and the corresponding transition matrix T which is 5×5 and nonregular but which does have a steady state to which all states converge and find this steady state.

Exercise 6.7. Consider the following population movement diagram:

(a) Write down the corresponding transition matrix T.

(b) If the population distribution starts at $\bar{x}_0 = \begin{bmatrix} 0.1 \\ 0 \\ 0.9 \end{bmatrix}$ what will it be after one iteration? How about after two iterations? How about after five iterations?

(c) Find the steady state distribution $\bar{x}^*$.

(d) Find the smallest value of k such that $T^k \bar{x}_0$ agrees with $\bar{x}^*$ to four decimal places.

Exercise 6.8. Find all eigenvalues of the following transition matrix:

$$T = \begin{bmatrix} a & 0 & 1-c \\ 1-a & b & 0 \\ 0 & 1-b & c \end{bmatrix}$$

Exercise 6.9. It seems like $T^k \bar{x}_0$ gets very close to $\bar{x}^*$ very quickly but this doesn't have to be the case. Find an example of a 2×2 regular transition matrix and an initial state $\bar{x}_0$ such that all of the entries in $\bar{x}_{1000} = T^{1000} \bar{x}_0$ still differ from those in $\bar{x}^*$ at the first decimal place.

Exercise 6.10. Give an example of a transition matrix whose corresponding population movement diagram is separated into two non-connected components.

Exercise 6.11. Give an example of a transition matrix whose corresponding population movement diagram is separated into two non-connected components, one of which has a steady-state to which all states converge and one of which does not.

Exercise 6.12. Define the following terms: Probability vector, transition matrix, regular transition matrix.

Exercise 6.13. For any given n is it possible to construct a transition matrix T such that some entry in T^k is 0 for $k = 1, ..., n-1$ but that entry is nonzero in T^n? Explain why or why not.

Exercise 6.14. Consider:
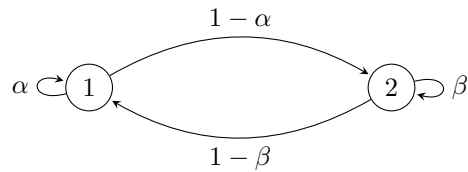
(a) Write down the corresponding transition matrix T.

(b) Assuming that $0 < \alpha < 1$ and $0 < \beta < 1$, find the limiting steady state vector for T.

(c) If remove the above restrictions on α and β does T have to have a limiting steady state vector? If it does not have to, can it? Justify.

Exercise 6.15. Suppose that T is regular and so T^k has all nonzero entries for some k. Explain why T^j has all nonzero entries for $j > k$.