1. Questions (a) and (b) are independent of one another.

(a) Write down the Google Pagerank matrix corresponding to the following internet. You can leave the matrix in expanded form. [10 pts]

(b) Suppose the 85/15 split for Google Pagerank were replaced by the following rule: If a page has outbound links then there is a 100% chance it will follow one of them at random. If a page has no outbound links it will jump randomly to some other page. Give an example of a nontrivial internet (it must have a few pages with at least a few links) for which the Pagerank would turn out to be 0 and explain why this would be the case. No complicated calculations are required if you justify in a quantitative manner. [10 pts]

2. Consider the following plot of 100 points. Assume the axis scale is 1 : 1.

Suppose that these points are placed together in a matrix $M$ with each point stored as a column and then the singular value decomposition is performed $M = U\Sigma V^T$.

(a) What are the dimensions of $U$, $\Sigma$ and $V$? [5 pts]

(b) Give a reasonable approximation for the first column of $U$. Justify. [5 pts]

(c) Give a reasonable approximation for the second column of $U$. Justify. [5 pts]

(d) There are two singular values. How would they compare to one another? Justify. [5 pts]

(e) Suppose you remove the smaller of the two singular values to get $\Sigma'$ and then recalculated $U\Sigma'V^T$. Treating each column as a point, draw a reasonable sketch of of the resulting points. Note: You don’t need to draw 100 points, just give the basic idea. [5 pts]

(f) Suppose you remove the larger of the two singular values to get $\Sigma'$ and then recalculated $U\Sigma'V^T$. Treating each column as a point, draw a reasonable sketch of of the resulting points. Note: You don’t need to draw 100 points, just give the basic idea. [5 pts]
3. Questions (a) and (b) are independent of one another.

(a) Suppose you have ten samples of the letter Q, each having resolution $16 \times 16$. Explain the process by which you would create a character basis matrix $B_Q$ for this letter using the two most significant singular values. [10 pts]

(b) Suppose your alphabet has two characters with resolution $2 \times 2$ for which you have calculated two character basis matrices as follows. These are approximated to one digit for simplicity.

$$B_1 = \begin{bmatrix} 0.7 & -0.6 \\ 0.4 & 0.6 \\ 0.1 & 0.4 \\ 0.6 & 0.3 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 0.1 & 0.7 \\ 0.8 & -0.1 \\ 0.1 & 0.7 \\ 0.6 & 0 \end{bmatrix}$$

Categorize the letter whose image has matrix:

$$\begin{bmatrix} 0.1 & 0.8 \\ 0.0 & 0.7 \end{bmatrix}$$

Note: There is a bunch of calculation involved. It’s all pretty manageable but you can earn most of the credit just by explaining what you would do.

4. Suppose a $100 \times 100$ image has singular values $s_1 \geq s_2 \geq ... \geq s_{100}$. The answers to the following will have $s_i$ in them since there are no actual values.

(a) How much image variance would be preserved if the image were compressed using only the largest ten singular values? [5 pts]

(b) Explain how you would calculate the minimum number of singular values necessary to preserve 99.9% of the image variance. [5 pts]

5. The following graph has Fielder vector given on the right:

(a) Use this vector to partition the graph into two subgraphs. You don’t need to draw any graphs, just explain which vertices go in which subgraphs. [5 pts]

(b) Use this vector to partition the graph into four subgraphs and use these subgraphs to draw a more reasonable picture of the graph. There’s some wiggle room for where you choose to partition so make sure you justify whichever choices you make. [10 pts]
6. Suppose you obtained the key fragment \( x_1, x_2, \ldots, x_{30} \) for a linearly recursively defined key.

   (a) If you found \( \det(M_m) \) for \( m = 1, 2, \ldots, 10 \) to be 0, 1, 1, 0, 0, 1, 0, 0, 0, 0 which value of \( m \) would you expect to be the key length? Why? \[5 \text{ pts}\]

   (b) Which corresponding matrix equation would you solve? This will have \( x_i \) in it! \[5 \text{ pts}\]

   (c) Suppose that this \( m \) did not work and you needed to calculate more determinants. What is the largest \( m \) you could calculate given the length of your key fragment? Explain. \[5 \text{ pts}\]

7. Suppose the covariance matrix associated to two stocks is given by:

\[
\begin{bmatrix}
0.040 & 0.002 \\
0.002 & 0.025
\end{bmatrix}
\]

Suppose the average returns of the stocks are \( \mu_1 = 0.01 \) and \( \mu_2 = 0.05 \) and suppose you allocate your portfolio with proportions \( x_1 \) in Stock 1 and \( x_2 \) in Stock 2.

   (a) Which matrix equation would you solve to find the global minimum variance portfolio? \[5 \text{ pts}\]

   (b) If your desired return is \( \mu = 0.025 \) which matrix equation would you solve to find the associated minimum variance portfolio? \[10 \text{ pts}\]

   (c) For an unknown desired return \( \mu \) the solution to the associated minimum variance portfolio matrix equation is given by:

\[
\begin{bmatrix}
1.25 - 25\mu \\
25\mu - 0.25 \\
2.725 - 76.25\mu \\
2.6625\mu - 0.1256
\end{bmatrix}
\]

For which values of \( \mu \) would you not have to short either stock?