1. Consider the matrix equation $A\bar{x} = \bar{b}$ given by:

$$
\begin{bmatrix}
1 & 0 \\
-1 & 1 \\
2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\bar{x} \\
\bar{b} \\
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
2 \\
3 \\
\end{bmatrix}
$$

(a) Find the least-squares solution the long way. This means explicitly find $\hat{\bar{b}} = \text{Proj}_{\text{Col}(A)} \bar{b}$ and then solve $A\hat{\bar{x}} = \hat{\bar{b}}$. If you’re not sure how to do the first part of this, ask!

(b) Find the least-squares solution by solving $A^TA\hat{\bar{x}} = A^T\bar{b}$ and check that the answers match.

2. Given the points $(2, 3), (10, 7), (15, 10)$, if we’re looking for a best-fit line it’s possible to look both for $y = mx + b$ and for $x = ny + c$. Neither has an exact solution but both have least-squares solution. Find each of these. Show that these don’t yield the same line. Plot the points and both lines. From a geometric perspective of minimizing distance from the line, what is going on here?

3. Suppose you would like to estimate the elliptical orbit of a certain object around the origin. Observations are made of both an angle and a distance. You collect five observations as follows where the first value is degrees (counterclockwise from the positive $x$-axis) and the second is in millions of miles:

$(23^\circ, 152), (50^\circ, 135), (100^\circ, 102), (110^\circ, 110), (152^\circ, 137)$

The equation of an ellipse in polar coordinates can be given by the following for some $A$ and $B$ where $\theta$ is the angle (counterclockwise from the positive $x$-axis) and $r$ is distance from the origin:

$$Ar^2\cos^2\theta + Br^2\sin^2\theta = 1$$

(a) Find the least-squares best-fit ellipse.

(b) Use this to predict the distance of the object when $\theta = 225^\circ$.

(c) What is the furthest that the object ever gets from the origin?

4. Consider the set of $n + 2$ points:

$$(1, 1), (2, 1), (3, 2), (3, 2), \ldots, (3, 2)$$

Suppose you wish to best-fit these to a line $y = mx + b$ using least-squares.

(a) Write down the corresponding matrix equation.

(b) Solve for $\hat{x}_n$ using the method of least squares. Make sure you simplify; the answer should not be complicated.

(c) Find $\lim_{n\to\infty} \hat{x}_n$.

(d) The line corresponding to your answer in (c) passes through $(3, 2)$. Why does this make sense?