1. Consider the following population movement diagram:

(a) Write down the corresponding transition matrix $T$.

(b) If the population distribution starts at $\bar{x}_0 = \begin{bmatrix} 0.1 \\ 0 \\ 0.9 \end{bmatrix}$ what will it be after one iteration? How about after two iterations? How about after five iterations?

(c) Find the steady state distribution $\bar{x}^*$.

(d) Find the smallest value of $k$ such that $T^k \bar{x}_0$ agrees with $\bar{x}^*$ to four decimal places.

2. Given the following population movement diagram:
Find each of the following without actually finding the transition matrix $T$.

(a) The $(2, 3)$ entry of $T^2$.
(b) The $(2, 5)$ entry of $T^3$.
(c) The smallest $k$ such that the $(1, 5)$ entry of $T^k$ is nonzero and what that value is.
(d) All $(i, j)$ such that the $(i, j)$ entry of $T^k$ is zero for all $k$.
(e) Explain intuitively what will happen in the long term to any initial population distribution. Justify intuitively. This question can be answered to various degrees of detail so say as much as you can.

3. Suppose a very simple version of English consists of five parts of speech: Noun (cat, dog, etc.), Verb (eat, sleep, etc.), Preposition (to, with, etc.), Article (a, the, etc.) and Conjunction (and, or). In any given sentence each of these has a probability of being followed by one of the others according to the following chart. To clarify, for example, the value 0.65 means that a Verb has a 0.65 probability of being followed by an Article.

<table>
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<th></th>
<th>N</th>
<th>V</th>
<th>P</th>
<th>A</th>
<th>C</th>
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<tr>
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<tr>
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</tbody>
</table>

(a) The sentence fragment “the cat eats the cheese” has the form ANVAN. Starting with A, calculate the associated probability of ANVAN appearing.
(b) The sentence fragment “dog to eat and with” has the form NPVC. Starting with N, calculate the associated probability of the form appearing.
(c) Compare the previous two values. What does their difference suggest?
(d) What is the probability that a Noun is followed two words later by an Article? Note: This means something like “Noun **** Article”.
(e) What is the probability that an Article is followed four words later by a Noun?
(f) Write down a diagram for this table where the nodes are N,V,P,A,C and an arrow from X to Y indicates the probability of X being followed by Y.
(g) Write down the corresponding transition matrix.
(h) The corresponding steady state can be interpreted to give the probabilities of the parts of speech appearing in the language. Calculate these probabilities.

4. It seems like $T^k\bar{x}_0$ gets very close to $\bar{x}_*$ very quickly but this doesn’t have to be the case. Find an example of a $2 \times 2$ regular transition matrix and an initial state $\bar{x}_0$ such that all of the entries in $T^{1000}\bar{x}_0$ still differ from those in $\bar{x}_*$ at the first decimal place.