Math 403 Exam 1 Sample 2 Hints

1. Use the Cayley table shown to answer the following:

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(a) Find and justify the identity element.
**Solution:** Which element \( e \) satisfies \( ey = ye = y \) for all other \( y \)?

(b) Find \( D^{-1} \).
**Solution:** Which element \( x \) satisfies \( xD = Dx = e \)?

(c) Is the group Abelian? Justify.
**Solution:** Can you find any \( x, y \) with \( xy \neq yx \)?

(d) Find and justify an element of order 3.
**Solution:** Find an \( x \) with \( x \neq e \), \( x^2 \neq e \) but \( x^3 = e \).

2. Classify each of the following as a group or not a group.

- For each which is a group give the identity and give an example of a nonidentity element and its inverse.
- For each which is not a group give an example of a single requirement which fails.

(a) \( \{1, 2, 3, 4, 5\}, \cdot \mod 6 \)
**Solution:** Not a group, there’s an element with no inverse.

(b) \( (\mathbb{R}^+, \cdot) \)
**Solution:** Not a group, the operation is not associative.

(c) \( (GL_2\mathbb{Q}, \cdot) \)
**Solution:** Group.

3. For each of the following, give \( \langle g \rangle \) for all \( g \in G \). Then determine if each group is cyclic and if so list all the generators.

(a) \( \mathbb{Z}_8 \)
**Solution:** List not give. Cyclic.

(b) \( U(14) \)
**Solution:** List not given. Cyclic.

4. Suppose \( G \) is cyclic with \( |G| = 36 \) and generator \( g \in G \).

(a) Find all subgroups of \( G \).
**Solution:** Use the Fundamental Theorem of Cyclic Groups. There’s one for each divisor.

(b) Find all generators of \( G \).
**Solution:** Generators are \( g^k \) with \( \gcd(36, k) = 1 \).

(c) What can you say about the number of elements of order 10 in \( G \)?
**Solution:** It’s \( \phi(10) \).

5. Prove that \( (\mathbb{R}^*, \cdot) \) is not cyclic.
**Solution:** Pick a non-identity element and examine its powers. Argue based on what you find.

6. Use the One-Step Subgroup Test to show that \( Z(G) \) is a subgroup of \( G \).
**Solution:** Straightforward.
7. (a) Suppose \( x, y \in G \) with \( xy \in Z(G) \). Show \( xy = yx \).
   **Solution:** Since \( xy \in Z(G) \) we have \( xy = yy^{-1}(xy) = y(xy)y^{-1} = yx \).

(b) Show that if \( |G| \) is even then \( G \) contains a nonidentity element of order 2.
   **Solution:** An element \( g \neq e \) satisfies \( g^2 = e \) iff \( g = g^{-1} \). Assume \( G \) has no elements of order 2. Then no element in \( G \) is its own inverse except for \( e \). This means nonidentity elements can be paired up with their inverses. However there are an odd number of nonidentity elements. This is a contradiction.

8. Suppose that \( G \) is Abelian and \( a, b \in G \) with \( |a| = |b| = 2 \). Show that \( G \) has a subgroup of order 4. Don’t just give the elements, justify the requirements for being a group.
   **Solution:** The elements are \( \{e, a, b, ab\} \). A Cayley table suffices.

9. Suppose some \( g \in G \) has the same order as all of its positive powers. Find \( g \). Justify.
   **Solution:** For all \( k \in \mathbb{Z}^+ \) we have \( |g| = |g^k| = |g|/\gcd(|g|, k) \) and so \( \gcd(|g|, k) = 1 \) for all such \( k \). Thus \( |g| = 1 \) and so \( g = e \).

10. Show that a group of order 4 must be Abelian.
    **Solution:** Put \( G = \{e, a, b, c\} \). Look at various possibilities of what happens when you combine elements. For example what could \( ab \) be? This can be a bit icky.

11. Two of the following are homomorphisms and two are not. Provide proof.
    
    (a) \( \phi : \mathbb{Z} \to \mathbb{Z} \) given by \( \phi(x) = 2x \).
    **Solution:** Homomorphism.

    (b) \( \phi : \mathbb{Z} \to \mathbb{Z}_5 \) given by \( \phi(x) = x + 1 \mod 5 \).
    **Solution:** Not a homomorphism.

    (c) \( \phi : \mathbb{R} \to \mathbb{C}^* \) given by \( \phi(x) = \cos x + i \sin x \).
    **Solution:** Homomorphism.

    (d) \( \phi : S_4 \to A_4 \) given by \( \phi(\alpha) = \begin{cases} 
    \alpha & \text{if } \alpha \in A_4 \\
    (12)\alpha & \text{if } \alpha \notin A_4
    \end{cases} \)
    **Solution:** Not a homomorphism.