MATH 403: Homework 1

1. For each of the following sets combined with an operation do the following. No explanation or proof is necessary:
   - State whether the set and operation form a group.
   - For each which forms a group give the identity and state if the group is Abelian or not.
   - For each which is not a group state which criteria (closure, associativity, identity or inverses) fail.

   (a) \((\mathbb{Z}, -)\)
   (b) \((\mathbb{Z}_5, \cdot \mod 5)\)
   (c) \((\mathbb{Z}_6, \cdot \mod 6)\)
   (d) \((\mathbb{Z}_6, + \mod 6)\)
   (e) \((\text{GL}_2\mathbb{R}, \cdot)\)
   (f) \((\text{GL}_2\mathbb{Z}, \cdot)\)
   (g) \(\{7, 14, 21, 28\}, \cdot \mod 35\)

2. In the group \(U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}, \cdot \mod 20\) find the inverse of each element.

3. The set \(\{5, 15, 25, 35\}\) is a group under multiplication modulo 40. Find and justify the identity element.

4. Let \(G\) be a group with a finite number of elements. Show that there must be an odd number of elements \(x \in G\) with the property that \(x^3 = e\).

5. Prove that a group \(G\) is Abelian iff \(\forall a, b \in G\) we have \((ab)^2 = a^2b^2\).

6. What is wrong with the following argument: In \(U(20)\) we have \(9^4 = 1\) and so \(|9| = 4\)?

7. Find the orders of all the elements in the following groups, no proof is necessary.
   (a) \(\mathbb{Z}_{12}\)
   (b) \(U(20)\)
   (c) \(\mathbb{R}^*\)

8. List the elements in each of the following subgroups.
   (a) \((8)\) in \(\mathbb{Z}_{14}\).
   (b) \((3)\) in \(U(20)\).
   (c) \(\left\langle \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \right\rangle\) in \(\text{GL}_2\mathbb{R}\) for some fixed \(\alpha \in \mathbb{R}\).

9. Let \(g = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \in \text{GL}_2\mathbb{R}\). Find/describe the elements in \(C(g)\).

10. Prove that \(Z(\text{GL}_2\mathbb{R})\) consists only of matrices of the form \(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\) with \(0 \neq a \in \mathbb{R}\). Note that you must show two things, first that matrices like this are in the center and second that everything in the center looks like one of these matrices.

11. Let \(G\) be a group. Use the One-Step Subgroup Test to prove that for \(a \in G\) the centralizer of \(a\) is a subgroup of \(G\).

12. Draw a Cayley table for \(D_4\).