MATH 403: Homework 10 (Chapter 21) Partial Solutions

1. Let $E$ be a field extension of $F$ and suppose $\alpha \in E$ is algebraic over $F$. Prove that $\alpha$ is a root of a unique irreducible monic polynomial in $F[x]$.

    **Solution:** This is corollary 3(c) in the Chapter 21 notes.

2. Find the degree and a basis for $Q(\sqrt[3]{3} + \sqrt[5]{5})$ over $Q(\sqrt{15})$.

    **Solution:** First note that $Q(\sqrt[3]{3} + \sqrt[5]{5}) = Q(\sqrt[3]{3}, \sqrt[5]{5})$ by a similar proof to HW9#6. Therefore every element in $Q(\sqrt[3]{3} + \sqrt[5]{5})$ has the form:

    \[ a + b\sqrt[3]{3} + c\sqrt[5]{5} + d\sqrt{15} = (a + d\sqrt{15})(1) + \left(b + c\frac{\sqrt{15}}{3}\right)(\sqrt[3]{3}) \]

    and thus a basis is \{1, $\sqrt[3]{3}$\} and the degree is 2.

3. For two subfields $F_1 \subseteq F$ and $F_2 \subseteq F$ define the composite field $F_1F_2$ to be the smallest subfield of $F$ containing both $F_1$ and $F_2$.

   (a) Show by example that $F_1F_2$ is not necessarily the same as $F_1 \cap F_2$.

       **Solution:** For example $R \subseteq \mathbb{R}$ and $Q \subseteq \mathbb{R}$ but $R \cap Q = Q$ and $RQ = \mathbb{R}$.

   (b) Show that $Q(\sqrt{2})Q(\sqrt[3]{3}) = Q(\sqrt{2}, \sqrt[3]{3})$.

       **Solution:** First observe that:

       \[ Q(\sqrt{2})Q(\sqrt[3]{3}) = Q(\sqrt{2}, \sqrt[3]{3}) \]

       since both the result of the the same process, thus we only need to show that:

       \[ Q(\sqrt{2}, \sqrt[3]{3}) = Q(\sqrt[3]{3}) \]

       Since $\sqrt{2}$ is a root $x^2 - 2$ which is irreducible over $Q$ and $\sqrt[3]{3}$ is a root of $x^3 - 2$ which is irreducible over $Q$ (by Eisenstein) an element in $Q(\sqrt{2}, \sqrt[3]{3})$ has the form:

       \[ a + b\sqrt[3]{3} + c\sqrt[5]{5} + d\sqrt{15} \]

       Since we have $2^{1/2} = (2^{1/6})^3$ and $2^{1/3} = (2^{1/6})^2$ we know that every element in $Q(\sqrt{2}, \sqrt[3]{3})$ is in $Q(\sqrt[3]{3})$.

       Since $\sqrt[3]{3}$ is a root of $x^6 - 2$ which is irreducible over $Q$ (by Eisenstein) en element in $Q(\sqrt[3]{3})$ has the form:

       \[ a + b\sqrt[1/2]{3} + c\sqrt[2/3]{2} + d\sqrt[3/6]{3} + e\sqrt[4/6]{4} + f\sqrt[5/6]{5} \]

       Since we have $2^{1/6} = 2^{1/2}/2^{1/3}$ we know that every element in $Q(\sqrt[3]{3})$ is in $Q(\sqrt[3]{3}, \sqrt[3]{3})$.

4. Find the minimal polynomial for $\sqrt[3]{2} + \sqrt[5]{5}$ over $Q$. Justify why it’s minimal.

   **Solution:** We can find the polynomial via:

   \[ x = \sqrt[3]{2} + \sqrt[5]{5} \]

   \[ x^3 = \left(\sqrt[3]{2} + \sqrt[5]{5}\right)^3 \]

   \[ x^3 = 2 + 3 \cdot 2^{2/3} \cdot 5^{1/3} + 3 \cdot 2^{1/3} \cdot 5^{2/3} + 5 \]

   \[ x^3 = 7 + 3 \cdot 2^{1/3} \cdot 5^{1/3} \left(2^{1/3} + 5^{1/3}\right) \]

   \[ x^3 - 7 = 3 \cdot 2^{1/3} \cdot 5^{1/3} x \]

   \[ (x^3 - 7)^3 = 3^3 \cdot 2^{1/3} \cdot 5^{1/3} x \]

   \[ x^9 - 21x^6 + 147x^3 - 343 = 27(2)(5)x^3 \]

   \[ x^9 - 21x^6 - 123x^3 - 343 = 0 \]

   I deleted the justification for why it’s minimal.
5. Find and justify \( \left[ \mathbb{Q} \left( \sqrt{2} + \sqrt{3} \right) : \mathbb{Q} \right] \).

**Solution:** We can find the polynomial via:
\[
\begin{align*}
x &= \sqrt{2} + \sqrt{3} \\
x^2 &= 2 + \sqrt{3} \\
x^2 - 2 &= \sqrt{3} \\
x^4 - 4x^2 + 4 &= 3 \\
x^4 - 4x^2 + 1 &= 0
\end{align*}
\]

6. If \( \alpha, \beta \in \mathbb{R} \) are transcendental over \( \mathbb{Q} \) prove that at least one of \( \alpha \beta \) and \( \alpha + \beta \) is also transcendental over \( \mathbb{Q} \).

**Solution:** Let \( K \) consist of all the elements in \( \mathbb{C} \) which are algebraic over \( \mathbb{Q} \).
Assume both \( \alpha \beta \) and \( \alpha + \beta \) are algebraic over \( \mathbb{Q} \) and consider that \( \alpha \) and \( \beta \) are roots of the polynomial
\[
(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha \beta \in K[x]
\]
Thus \( \alpha \) and \( \beta \) are algebraic over \( K \) and hence since \( K \) is algebraic over \( \mathbb{Q} \) both \( \alpha \) and \( \beta \) are algebraic over \( \mathbb{Q} \), a contradiction.

**Definition for Questions 7,8:** A field \( F \) is **algebraically closed** if every nonconstant polynomial in \( F[x] \) has a root in \( F \).

7. Prove that if \( F \) is finite then \( F \) is not algebraically closed.

**Solution:** If we have \( F = \{a_1, \ldots, a_n\} \) then consider that the polynomial \( (x - a_1)(x - a_2) \ldots (x - a_n) + 1 \in F[x] \) but has no roots in \( F \).

8. It’s true that \( \mathbb{C} \) is algebraically closed. Given this fact, show that if \( E \) is a finite extension of \( \mathbb{R} \) then \( E \approx \mathbb{C} \) or \( E \approx \mathbb{R} \).

**Solution:** Deleted. Do not grade.