3. Find the smallest \( n \) such that \( U(n) \) is not cyclic. You don’t need to prove it’s the smallest but you do need to prove it’s not cyclic.

**Proof:** \( U(8) \) is not cyclic because \( U(8) = \{1, 3, 5, 7\} \) and \(|1| = 1\) and \(|3| = |5| = |7| = 2\).

5. Suppose \( G \) is a group and \( g \in G \) has \(|g^4| = 12\). What could \(|g|\) be? Prove.

**Solution:** Let \( n = |g|\). We know that \( 12 = |g^4| = \frac{n}{\gcd(n, 4)} \). Observe that because the gcd must divide 4 we could have \( \gcd(n, 4) = 1, 2, 4 \).

If \( \gcd(n, 4) = 1 \) then the equation becomes \( 12 = \frac{n}{4} \) so \( n = 12 \) but \( \gcd(12, 4) \neq 1 \), a contradiction.

If \( \gcd(n, 4) = 2 \) then the equation becomes \( 12 = \frac{n}{2} \) so \( n = 24 \) but \( \gcd(24, 4) \neq 2 \), a contradiction.

If \( \gcd(n, 4) = 4 \) then the equation becomes \( 12 = \frac{n}{4} \) so \( n = 48 \) and \( \gcd(48, 4) = 4 \). Thus we must have \(|g| = 48\).

8. Prove that a finite group is the union of its proper subgroups if and only if the group is not cyclic.

**Proof:**

\( \Rightarrow \) Suppose a finite group \( G \) is the union of its proper subgroups. Assume \( G \) is cyclic and let \( g \) be a generator. Then since \( G \) is the union of its proper subgroups \( g \) must be in one of those proper subgroups. However this means that \( G = \langle g \rangle \) must be in that proper subgroup, contradicting the fact that it’s proper. Thus \( G \) is not cyclic.

\( \Leftarrow \) Suppose \( G \) is not cyclic. Then \( G \) is the union of all \( \langle a \rangle \) with \( a \in G \). Since none of these are all of \( G \) they are all proper. Note that there may be more proper subgroups but they we’ve already got all of \( G \) so they don’t contribute anything new.