1. Determine whether each of the following subsets is an ideal. For those which are, no proof is required. For those which aren’t, give explicit \( r \in R \) and \( a \in A \) with \( ra \notin A \).

(a) \( 3 \mathbb{Z} \subset \mathbb{Z} \)
(b) \( \mathbb{Z} \subset \mathbb{R} \)
(c) \( \{ x^2 p(x) \mid p(x) \in \mathbb{Z}[x] \} \subset \mathbb{Z}[x] \)
(d) \( \text{GL}_2 \mathbb{R} \subset M_2 \mathbb{R} \)
(e) \( \mathbb{Z}[x] \subset \mathbb{R}[x] \)

2. For each of the following ideals and elements, determine if each element is in the ideal. Justify briefly.

(a) The ideal \( \langle x \rangle \) in the ring \( \mathbb{Z}[x] \): The elements \( 3x + 1, 4x^2 - 2x \) and \( \frac{1}{2}x^2 \).
(b) The ideal \( \langle x^2 + 3 \rangle \) in the ring \( \mathbb{R}[x] \): The elements \( x^3 + 3x, x^4 - x^3 + 4x^2 - 3x + 3 \) and \( x^4 - x^2 + x - 13 \).
(c) The ideal \( \langle x, 4 \rangle \) in the ring \( \mathbb{R}[x] \): The elements \( x^2 + x \) and \( 2x^2 - x + 8 \).
(d) The ideal \( \langle 5x + 6 \rangle \) in the ring \( \mathbb{Z}_7[x] \): The elements \( x^2 + x + 2 \) and \( x^2 + 1 \).

3. Consider the ring \( \mathbb{Z}[x] \) and the ideal \( \langle x^2 - 3 \rangle \).

(a) Describe the elements in \( \mathbb{Z}[x]/\langle x^2 - 3 \rangle \) as simply as possible, with justification.
(b) Calculate \( (2x + 2 + \langle x^2 - 3 \rangle) + (-5x - 1 + \langle x^2 - 3 \rangle) \), simplifying the answer to the form you found in (a).
(c) Calculate \( (4x - 3 + \langle x^2 - 3 \rangle) (3x - 3 + \langle x^2 - 3 \rangle) \), simplifying the answer to the form you found in (a).

4. Show that \( A = \{(3x, y) \mid x, y \in \mathbb{Z} \} \) is a maximal ideal of \( \mathbb{Z} \oplus \mathbb{Z} \). Show both that it’s an ideal and that it’s maximal.

5. (a) Show that \( \langle 2 + 2i \rangle \) is not a prime ideal of \( \mathbb{Z}[i] \). You must prove any claims you make about whether elements are in or not in the ideal.
(b) Show that \( 26 \mathbb{Z} \) is a prime ideal of \( 2 \mathbb{Z} \). You may assume it’s an ideal and just show it’s prime.

6. Show that \( \mathbb{R}[x]/\langle x^2 + 1 \rangle \) is a field by showing that each nonzero element is a unit.

7. Suppose \( \phi \) is a 1-1 ring homomorphism from \( \mathbb{Z} \oplus \mathbb{Z} \) to itself. What are the possibilities for \( \phi(1, 0) \)? Justify.

8. Find rings \( R \) and \( S \), an ideal \( A \) of \( R \) and a ring homomorphism \( \phi : R \to S \) such that \( \phi(A) \) is not an ideal of \( S \). You must prove that \( A \) is an ideal, that \( \phi \) is a ring homomorphism and that \( \phi(A) \) is not an ideal of \( S \).

9. (a) Show by example that \( \phi(x) : \mathbb{Z}_m \to \mathbb{Z}_n \) given by \( \phi(x) = ax \) (for \( a \in \mathbb{Z} \)) need not be a ring homomorphism.
(b) Show that \( \phi : \mathbb{Z}_5 \to \mathbb{Z}_{30} \) given by \( \phi(x) = 6x \) is a ring homomorphism.

10. Prove that \( \mathbb{C} \not\approx \mathbb{R} \) as rings.

11. Prove that \( \mathbb{Z}[\sqrt{-2}] \approx \mathbb{Z}[x]/\langle x^2 + 2 \rangle \) as rings.

12. Show that a homomorphism from a field onto a ring with more than one element must be an isomorphism.