Math 403 Chapter 15: Ring Homomorphisms

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1. **Introduction:** As with groups, among other things, ring homomorphism are a way of creating ideals. In reality we'll use them less than we did with groups.

2. **Homomorphisms - Basics:**

   (a) **Definition:** A *ring homomorphism* from a ring R_1 to a ring R_2 is a mapping $\phi : R_1 \to R_2$ such that for all $a, b \in R_1$ we have:

   - $\phi(a + b) = \phi(a) + \phi(b)$
   - $\phi(ab) = \phi(a)\phi(b)$

   Note that the operations may in theory differ, the left being in R and the right in S.

   Also note that a ring homomorphism is in fact a group homomorphism with the group operation being the $+$ inside the ring.

   (b) **Definition:** A ring homomorphism is a *ring isomorphism* if it is 1-1 and onto.

   (c) **Definition:** The kernel of a ring homomorphism $\phi : R_1 \to R_2$ is the set:

   $$\text{Ker}\phi = \{a \in R_1 \,|\, \phi(a) = 0 \in R_2\}$$

   (d) **Examples:**

   Example: The mapping $\phi : \mathbb{Z} \to \mathbb{Z}_n$ given by $\phi(x) = x \bmod n$ is a ring homomorphism. The kernel is $n\mathbb{Z}$.

   Example: The mapping $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(a + bi) = a - bi$ is a ring homomorphism. The kernel is 0.

   Example: The mapping $\phi : \mathbb{R}[x] \to \mathbb{R}$ given by $\phi(p(x)) = p(2)$ is a ring homomorphism. The kernel is all the polynomials with x-intercept at $x = 2$.

   Example: The mapping $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_{30}$ given by $\phi(x) = 10x \bmod 30$ is a ring homomorphism. This is not obvious because of the modulus change. To clarify the problem, when we write $\phi(x) = 10x \bmod 30$ we assume $x \in \{0, 1, ..., 11\}$. However when we do $\phi(x + y)$ we have $x, y \in \{0, 1, ..., 11\}$ but $x + y$ is not necessarily so, rather it is reduced mod 12 and then ϕ is applied. Thus what we are really trying to show is that:

   $$\phi((x + y) \bmod 12) = (\phi(x) \bmod 30) + (\phi(y) \bmod 30)$$

   To show this note that if we write $x + y = 12q + r$ with $0 \leq r < 12$ then:

   $$\begin{aligned}
   \phi((x + y) \bmod 12) &= \phi((12q + r) \bmod 12) \bmod 30 \\
   &= \phi(r) \bmod 30 \\
   &= 10r \bmod 30 \\
   &= 10(x + y - 12q) \bmod 30 \\
   &= 10x + 10y - 120q \bmod 30 \\
   &= 10x + 10y \bmod 30 \\
   &= (\phi(x) + \phi(y)) \bmod 30 \\
   &= (\phi(x) \bmod 30) + (\phi(y) \bmod 30)
   \end{aligned}$$

And similarly if we write $xy = 12q + r$ with $0 \leq r < 12$ then:

$$\begin{aligned}
\phi((xy) \bmod 12) &= \phi((12q + r) \bmod 12) \bmod 30 \\
&= \phi(r) \bmod 30 \\
&= 10r \bmod 30 \\
&= 10(xy - 12q) \bmod 30 \\
&= 10xy - 120q \bmod 30 \\
&= 10xy \bmod 30 \\
&= 10xy + 90xy \bmod 30 \\
&= (10x)(10y) \bmod 30 \\
&= (\phi(x)\phi(y)) \bmod 30 \\
&= (\phi(x) \bmod 30)(\phi(y) \bmod 30)
\end{aligned}$$

3. **Theorem (Properties):** Let $\phi : R \to S$ be a ring homomorphism. Let A be a subring of R and let B be an ideal of S.

   (a) $\phi(A)$ is a subring of S.

   (b) If A is an ideal of R then $\phi(A)$ is an ideal of $\phi(R)$. Thus if ϕ is onto then $\phi(A)$ is an ideal of S.

   (c) $\phi^{-1}(B)$ is an ideal of R.

   (d) If R is commutative then so is $\phi(R)$.

   (e) If R has unity 1, if $S \neq \{0\}$ and if ϕ is onto, then $\phi(1)$ is the unity for S.

   (f) ϕ is an isomorphism iff ϕ is onto and $\text{Ker}\phi = \{0\}$.

   (g) If ϕ is an isomorphism then $\phi^{-1} : S \to R$ is also an isomorphism.

   Proof: All are straightforward. $\mathcal{QED}$

4. **Connection to Quotient Rings**

   (a) **Theorem (Kernels are Ideals):** Let $\phi : R \to S$ be a ring homomorphism. Then $\text{Ker}\phi$ is an ideal of R.
   Proof: Straightfoward. $\mathcal{QED}$

   (b) **Theorem (First Isomorphism Theorem for Rings):** Let $\phi : R \to S$ be a ring homomorphism. Then the mapping $\psi : R/\text{Ker}\phi \to \phi(R)$ given by $\psi(r + \text{Ker}\phi) = \phi(r)$ is a ring isomorphism.
   Proof: Straightforward. $\mathcal{QED}$

   (c) **Theorem (Ideals are Kernels):** Every ideal A of a ring R is the kernel of the ring homomorphism from R to R/A taking $r \mapsto r + A$.
   Proof: Straightforward. $\mathcal{QED}$