1. **Introduction:** As with groups, among other things, ring homomorphisms are a way of creating ideals. In reality we’ll use them less than we did with groups.

2. **Homomorphisms - Basics:**

   (a) **Definition:** A ring homomorphism from a ring $R_1$ to a ring $R_2$ is a mapping $\phi : R_1 \rightarrow R_2$ such that for all $a, b \in R_1$ we have:
   
   $\begin{align*}
   \phi(a + b) &= \phi(a) + \phi(b) \\
   \phi(ab) &= \phi(a)\phi(b)
   \end{align*}$

   Note that the operations may in theory differ, the left being in $R$ and the right in $S$.

   Also note that a ring homomorphism is in fact a group homomorphism with the group operation being the $+$ inside the ring.

   (b) **Definition:** A ring homomorphism is a ring isomorphism if it is 1-1 and onto.

   (c) **Definition:** The kernel of a ring homomorphism $\phi : R_1 \rightarrow R_2$ is the set:
   
   $\text{Ker}\phi = \{ a \in R_1 | \phi(a) = 0 \in R_2 \}$

   (d) **Examples:**

   **Example:** The mapping $\phi : \mathbb{Z} \rightarrow \mathbb{Z}_n$ given by $\phi(x) = x \mod n$ is a ring homomorphism. The kernel is $n\mathbb{Z}$.

   **Example:** The mapping $\phi : \mathbb{C} \rightarrow \mathbb{C}$ given by $\phi(a + bi) = a - bi$ is a ring homomorphism. The kernel is 0.

   **Example:** The mapping $\phi : \mathbb{R}[x] \rightarrow \mathbb{R}$ given by $\phi(p(x)) = p(2)$ is a ring homomorphism. The kernel is all the polynomials with $x$-intercept at $x = 2$.

   **Example:** The mapping $\phi : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{30}$ given by $\phi(x) = 10x \mod 30$ is a ring homomorphism. This is not obvious because of the modulus change. To clarify the problem, when we write $\phi(x) = 10x \mod 30$ we assume $x \in \{0, 1, \ldots, 11\}$. However when we do $\phi(x + y)$ we have $x, y \in \{0, 1, \ldots, 11\}$ but $x + y$ is not necessarily so, rather it is reduced mod 12 and then $\phi$ is applied. Thus what we are really trying to show is that:

   $$\phi((x + y) \mod 12) = (\phi(x) \mod 30) + (\phi(y) \mod 30)$$

   To show this note that if we write $x + y = 12q + r$ with $0 \leq r < 12$ then:

   $$\begin{align*}
   \phi((x + y) \mod 12) &= \phi((12q + r) \mod 12) \mod 30 \\
   &= \phi(r) \mod 30 \\
   &= 10r \mod 30 \\
   &= 10(x + y - 12q) \mod 30 \\
   &= 10x + 10y - 120q \mod 30 \\
   &= 10x + 10y \mod 30 \\
   &= (\phi(x) + \phi(y)) \mod 30 \\
   &= (\phi(x) \mod 30) + (\phi(y) \mod 30)
   \end{align*}$$
And similarly if we write $xy = 12q + r$ with $0 \leq r < 12$ then:

\[
\phi((xy) \mod 12) = \phi((12q + r) \mod 12) \mod 30 \\
= \phi(r) \mod 30 \\
= 10r \mod 30 \\
= 10(xy - 12q) \mod 30 \\
= 10xy - 120q \mod 30 \\
= 10xy \mod 30 \\
= 10xy + 90xy \mod 30 \\
= (10x)(10y) \mod 30 \\
= (\phi(x)\phi(y)) \mod 30 \\
= (\phi(x) \mod 30)(\phi(y) \mod 30)
\]

3. **Theorem (Properties):** Let $\phi : R \to S$ be a ring homomorphism. Let $A$ be a subring of $R$ and let $B$ be an ideal of $S$.

(a) $\phi(A)$ is a subring of $S$.

(b) If $A$ is an ideal of $R$ then $\phi(A)$ is an ideal of $\phi(R)$. Thus if $\phi$ is onto then $\phi(A)$ is an ideal of $S$.

(c) $\phi^{-1}(B)$ is an ideal of $R$.

(d) If $R$ is commutative then so is $\phi(R)$.

(e) If $R$ has unity 1, if $S \neq \{0\}$ and if $\phi$ is onto, then $\phi(1)$ is the unity for $S$.

(f) $\phi$ is an isomorphism iff $\phi$ is onto and $\text{Ker}\phi = \{0\}$.

(g) If $\phi$ is an isomorphism then $\phi^{-1} : S \to R$ is also an isomorphism.

**Proof:** All are straightforward. $\text{QED}$

4. **Connection to Quotient Rings**

(a) **Theorem (Kernels are Ideals):** Let $\phi : R \to S$ be a ring homomorphism. Then $\text{Ker}\phi$ is an ideal of $R$.

**Proof:** Straightforward. $\text{QED}$

(b) **Theorem (First Isomorphism Theorem for Rings):** Let $\phi : R \to S$ be a ring homomorphism. Then the mapping $\psi : R/\text{Ker}\phi \to \phi(R)$ given by $\psi(r + \text{Ker}\phi) = \phi(r)$ is a ring isomorphism.

**Proof:** Straightforward. $\text{QED}$

(c) **Theorem (Ideals are Kernels):** Every ideal $A$ of a ring $R$ is the kernel of the ring homomorphism from $R$ to $R/A$ taking $r \mapsto r + A$.

**Proof:** Straightforward. $\text{QED}$