1. **Introduction**: An extremely basic notion of a group is a collection of objects and a way to combine them. There is of course a more formal definition as well as requirements but before nailing down the specifics here are some examples:

- **Example**: We could take the integers with addition. If we add two integers we get another integer.
- **Example**: We could take the various ways to switch the objects in three boxes with the notion of doing one switch and then another. If we do two switches the result is a switch.

2. **Definition(s)**: A group \( G \) is a set of objects (sometimes also sloppily denoted \( G \)) and a binary operation \( * \) (not necessarily multiplication) which takes two objects in \( G \), say \( a \) and \( b \), and creates a new object \( a * b \) which is also in \( G \) (this is called closure). Moreover we must have:

   (a) **Associativity**: For any \( a, b, c \in G \) we have \((a * b) * c = a * (b * c)\).

   (b) **Identity**: There is some \( e \in G \) such that for all \( a \in G \) we have \( e * a = a * e = a \). There is no assumption that this is unique!

   (c) **Inverses**: For every \( a \in G \) there is some \( b \in G \) with \( a * b = b * a = e \). There is no assumption that this is unique!

A word on notation. Often in the abstract we write \( ab \) instead of \( a * b \) and this is usually fine, especially when \( * \) is actually multiplication or something unambiguous. However if \( * \) is addition then we should write \( a + b \) instead of \( ab \). When we do use \( ab \) notation then sometimes instead of \( e \) we write \( 1 \) but this only sometimes makes sense.

3. **Abelian Groups**: Note that there is no guarantee that \( a * b = b * a \) for all \( a, b \in G \). When this is true we say the group is **Abelian**, or **commutative**.

4. **Examples and Non-Examples**: Here are some examples and non-examples:

   - **Example**: The structure \( G = (\mathbb{Z}, +) \) is an Abelian group.
   - **Example**: The structure \( (\mathbb{Z}, -) \) is not a group. Why not?
   - **Example**: The structure \( G = (\{1, 3, 5, 7\}, \cdot \mod 8) \) is an Abelian group.
   - **Example**: The structure \( G = (GL_2 \mathbb{R}, \cdot) \) is a group but is not Abelian.
   - **Example**: The structure \( (\mathbb{R}, \cdot) \) is not a group.
   - **Example**: The structure \( G = (\mathbb{R} - \{0\}, \cdot) \) is an Abelian group.

5. **Elementary Properties**: The following are properties of a group. Notice that they’re not part of the definition, rather they follow automatically from the definition.

   (a) **Theorem**: The identity is unique.

      **Proof**: Suppose \( e_1, e_2 \) are both identities. Then \( e_1 e_2 = e_1 \) and \( e_1 e_2 = e_2 \) so then \( e_1 = e_2 \).

   (b) **Theorem**: The left and right cancellation laws hold.

      **Proof**: Suppose \( ab = ac \). Left multiply by an inverse of \( a \). Note that sometimes this is stated for non-identity \( a \) but it’s fine for \( a = e \) too, the point being that if \( a = e \) then \( ab = ac \) becomes \( b = c \) without any cancellation at all.

   (c) **Theorem**: Inverses are unique.

      **Proof**: Suppose \( b_1, b_2 \) are both inverses of \( a \). Then \( ab_1 = e = ab_2 \) then cancel the \( a \).

      **Note**: Now we can use \( a^{-1} \) for the inverse of \( a \).

   (d) **Theorem**: The shoes-socks property holds.

      **Proof**: We wish to solve \((ab)(?) = e\). Note \( abb^{-1}a^{-1} = e \).