Math 403 Chapter 21: Algebraic Extensions

1. **Introduction:** Extension fields may be categorized several different ways. In this chapter we will look at some of these divisions.

2. **Algebraic v Transcendental**

   (a) **Definition:** Given an extension field E of F and an element $a \in E$, we say that a is *algebraic over* F if a is the root of a polynomial in F. Otherwise we say it is *transcendental over* F.

   Note: The base field F is important here. Often when people just say "transcendental" they mean over $\mathbb{Q}$ but that isn't the only possibility.

   Example: $\sqrt{2} \in \mathbb{R} \supseteq \mathbb{Z}$ is algebraic over $\mathbb{Z}$ because it is a root of the polynomial $x^2 - 2 \in \mathbb{Z}[x]$.

   Example: $\sqrt{2+\sqrt{3}} \in \mathbb{R} \supseteq \mathbb{Z}$ is algebraic over $\mathbb{Z}$ because it is a root of the polynomial $x^4 - 4x^2 + 1 \in \mathbb{Z}[x]$.

   Example: $\pi \in \mathbb{R} \supseteq \mathbb{Z}$ is transcendental over $\mathbb{Z}$ because there is no polynomial in $\mathbb{Z}[x]$ for which π is a root. This is hard to prove.

   Example: $\pi \in \mathbb{C} \supset \mathbb{R}$ is algebraic over $\mathbb{R}$ because it is a root of the polynomial $x-\pi \in \mathbb{R}[x]$.

   (b) **Definition:** An extension field E of F is called *an algebraic extension of* F if every element of E is algebraic over F. Otherwise we say it is *a transcendental extension of* F.

   (c) **Definition:** An extension field of the form $F(a)$ is a *simple extension of* F.

3. **Algebraic Extensions**

   (a) **Introduction:** Here we will focus specifically on a theorem related to algebraic extensions. It basically revisits something we know but from an opposite direction.

   (b) **Theorem:** Let E be an extension field of F and let $a \in E$. If a is algebraic over F then $F(a) \approx F[x]/\langle p(x)\rangle$ where $p(x)$ is a polynomial in $F[x]$ of minimal degree for which $p(a) = 0$. In addition such a $p(x)$ will be irreducible over F.

   Note: This isomorphism arose earlier in the FTOFT but in that case we started with an irreducible polynomial and constructed an extension field in which a root existed whereas in this case we starting with an extension field that we know about and a root in that extension field and an irreducible polynomial emerges.

   Proof: If a is algebraic over F then define $\phi : F[x] \to F(a)$ by $\phi(f(x)) = f(a)$. By the First Isomorphism Theorem we know that

   $$F[x]/\operatorname{Ker}\phi \approx \phi(F[x]) \subseteq F(a)$$

   Since a is algebraic over F there are $f(x) \in F[x]$ with $f(a) = 0$ and so $\operatorname{Ker}\phi \neq 0$. Thus we know that $\operatorname{Ker}\phi$ is a nonzero ideal of $F[x]$ which can be written in the form $\langle p(x)\rangle$ (since $F[x]$ is a PID) where $p(x)$ is a polynomial of minimal degree in the ideal (previous theorem). Since $p(x)$ has minimal degree it must be irreducible since if we could reduce $p(x) = f(x)g(x)$ then $0 = p(a) = f(a)g(a)$ would imply a polynomial of lower degree for which a were a root.

   Now then since $p(x)$ is irreducible we know that $\langle p(x)\rangle$ is maximal (previous theorem) and hence $F[x]/\langle p(x)\rangle$ is a field (prevous theorem) and since $F[x]/\langle p(x)\rangle \approx \phi(F[x])$ we know that $\phi(F[x])$ is a subfield of $F(a)$ containing both F (since $\phi(c) = c$ for $c \in F$) and a (since $\phi(x) = a$). But $F(a)$ is the smallest such subfield and so $\phi(F[x]) = F(a)$ and the result follows. $\mathcal{QED}$

Example: Consider $\sqrt{2} \in \mathbb{R} \supseteq \mathbb{Q}$ is algebraic over rationals, since it's a root of, among other things, $x^2 - 2 \in \mathbb{Q}[x]$, and so $\mathbb{Q}(\sqrt{2}) \approx F[x]/\langle p(x)\rangle$ where $p(x)$ is a polynomial in $\mathbb{Q}[x]$ of minimal degree which is irreducible over $\mathbb{Q}$. In fact we know that $\mathbb{Q}(\sqrt{2}) \approx F[x]/\left\langle x^2 - 2\right\rangle$ but in this new theorem we started with $\sqrt{2}$ and the theorem proves the existence of the polynomial from the field extension rather than the other way around.

(c) **Corollary:** If $a \in E \supseteq F$ is algebraic over F then there is a unique monic irreducible polynomial in $F[x]$ for which a is a root.

Proof: If $p(x)$ is the polynomial arising in the previous proof then we can multiply by the multiplicative inverse of the leading coefficient to get a monic irreducible polynomial. To show it is unique suppose $p_1(x) \neq p_2(x)$ were both monic irreducible polynomials of minimal degree with $p_1(a) = p_2(a) = 0$. Then $(p_1 - p_2)(x)$ would be a nonzero polynomial of smaller degree for which a is a root. Now then either $p_1 - p_2$ itself is irreducible or it has an irreducible factor which will also have a as a root. Either way we have a nonzero polynomial of smaller degree for which a is a root, a contradiction. $\mathcal{QED}$

(d) **Definition:** The polynomial arising in the previous theorem is called the *minimal polynomial for a over F*.

(e) **Corollary:** If $p(x)$ is the minimal polynomial for $a \in E \supseteq F$ over F then for all $f(x) \in F[x]$ with $f(a) = 0$ we have $p(x) \mid f(x)$ in $F[x]$.

Proof: For any other $f(x) \in F[x]$ with $f(a) = 0$ we know that $f(x) \in \operatorname{Ker}\phi = \langle p(x)\rangle$ with the ϕ from the theorem. Then $p(x) \mid f(x)$ by definition of $\langle p(x)\rangle$. $\mathcal{QED}$

4. **The Degree of an Extension**

(a) **Definition:** Let E be an extension field of F. We say that E *has degree n over F* and write $[E : F] = n$ if E has dimension n as a vector space over F. If $[E : F]$ is finite we say that *E is a finite extension of F* and otherwise we say that *E is an infinite extension of F*.

Note: Basically (haha) if we can find a set $B = \{b_1, ..., b_n\}$ taken from E such that every element of E can be written uniquely as a linear combination of elements of B using coefficients in E then B is the basis and $[E : F] = n$ is the dimension.

Example: We have $[\mathbb{C} : \mathbb{R}] = 2$ since $\{1, i\}$ form a basis for $\mathbb{C}$ over reals because every element of $\mathbb{C}$ can be written in the form $a(1) + b(i)$ with $a, b \in \mathbb{R}$.

Example: We have $\left[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}\right] = 3$ because elements in $\mathbb{Q}(\sqrt[3]{2})$ have the unique form $c_0 + c_1\sqrt[3]{2} + a_2(\sqrt[3]{2})^2$ with $c_0, c_1, c_2 \in \mathbb{Q}$ by a previous theorem.

(b) **Theorem:** If E is a finite extension of F then each $a \in E$ is algebraic over F and so E is algebraic over F.

Proof: Suppose $[E : F] = n$ and $a \in E$. The set $\{1, a, ..., a^n\}$ contains more than n elements and hence is linearly dependent over F, meaning there are constants $c_0, ..., c_n$ with $c_0 + c_1 a + ... + c_n a^n = 0$. Then a is a root of $f(x) = c_0 + c_1 x + ... + c_n x^n$ and hence is algebraic. $\mathcal{QED}$

Note: The converse is false, for example $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, ...)$ (forever!) is algebraic but not finite. Do you see why?

(c) **Theorem:** If we have finite field extensions $F \subseteq E \subseteq K$ then $[K : F] = [K : E][E : F]$.

Proof: Omit. The details are just icky and unenlightening and the basic idea can be captured with an example. $\mathcal{QED}$

Example: Suppose we take $\mathbb{Q}$ and extend it to $\mathbb{Q}(\sqrt{2})$ we have a degree 2 field extension with basis $\{1, \sqrt{2}\}$ in which all elements have the form $a + b\sqrt{2}$ with $a, b \in \mathbb{Q}$.

Suppose we then extend from $\mathbb{Q}(\sqrt{2})$ to $\mathbb{Q}(\sqrt{2})(\sqrt[3]{5}) = \mathbb{Q}(\sqrt{2}, \sqrt[3]{5})$. This is a degree 3 field extension with basis $\{1, \sqrt{5}, (\sqrt{5})^2\}$ in which all elements have the form $c + d\sqrt{5} + e(\sqrt{5})^2$ with $c, d, e \in \mathbb{Q}(\sqrt{2})$.

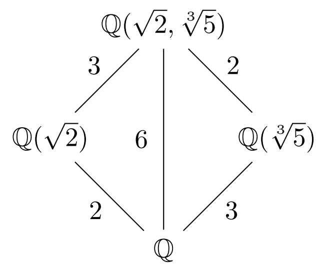
Really then all elements in $\mathbb{Q}(\sqrt{2}, \sqrt[3]{5})$ have the form:

$$\begin{aligned} c + d\sqrt{5} + e(\sqrt{5})^2 &= (a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})\sqrt[3]{5} + (a_3 + b_3\sqrt{2})(\sqrt[3]{5})^2 \\ &= a_1 + b_1\sqrt{2} + a_2\sqrt[3]{5} + b_2\sqrt{2}\sqrt[3]{5} + a_3(\sqrt[3]{5})^2 + b_3\sqrt{2}(\sqrt[3]{5})^2 \end{aligned}$$

Thus $\mathbb{Q}(\sqrt{2}, \sqrt[3]{5})$ is a degree 6 field extension of $\mathbb{Q}$ with basis:

$$\left\{1, \sqrt{2}, \sqrt[3]{5}, \sqrt{2}\sqrt[3]{5}, (\sqrt[3]{5})^2, \sqrt{2}(\sqrt[3]{5})^2\right\}$$

Note that conceptually we could have extended it to $\mathbb{Q}(\sqrt[3]{5})$ first, and this leads to the following diagram:



(d) **Note:** The theorem can also inform us about what field extensions are possible and whether elements are or are not in field extensions.

Example: By the above example any field extension between $\mathbb{Q}$ and $\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{5}\right)$ must have degree over $\mathbb{Q}$ which divides 6. This also tells us, for example, that $\sqrt[4]{7} \notin \mathbb{Q}\left(\sqrt{2}, \sqrt[3]{5}\right)$. This is because if it were then we would have:

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[4]{7}) \subseteq \mathbb{Q}\left(\sqrt{2}, \sqrt[3]{5}\right)$$

and hence:

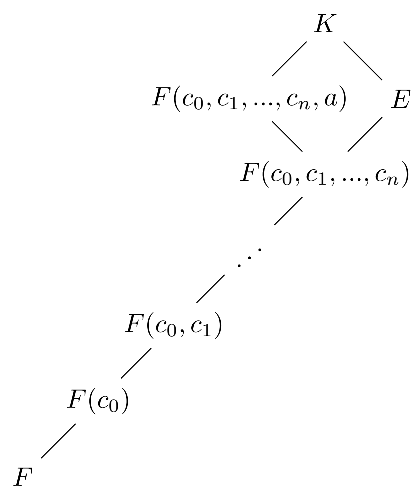
$$\underbrace{\left[\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{5}\right) : \mathbb{Q}\right]}_{6} = \underbrace{\left[\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{5}\right) : \mathbb{Q}(\sqrt[4]{7})\right]}_{?} \underbrace{\left[\mathbb{Q}(\sqrt[4]{7}) : \mathbb{Q}\right]}_{4}$$

However $4 \nmid 6$.

5. **Final Theorems**

(a) **Theorem:** If K is algebraic over E and E is algebraic over F then K is algebraic over F.

Proof: Let $a \in K$. Since K is algebraic over E there is some irreducible polynomial $p(x) = c_n x^n + ... + c_1 x + c_0$ with $c_i \in E$ such that $p(a) = 0$. Consider now the diagram:



Since each c_i is algebraic over F each field extension up until the split is finite. Moreover the left branch is degree n and so $a \in F(c_0, c_1, ..., c_n, a)$ which is a finite extension over F. Thus a is algebraic over F. $\mathcal{QED}$

(b) **Theorem:** Let E be an extension field of F. Then the set of all elements in E which are algebraic over F form a subfield of E.

Proof: Suppose $a, b \in E$ are algebraic over F and $b \neq 0$. Consider that $[F(a,b) : F] = [F(a,b) : F(b)][F(b) : F]$ which is finite since a, b are algebraic. Thus since $a+b, a-b, ab, a/b \in F(a,b)$ we know that all four are in a finite extension of F and hence are algebraic over F. Thus the set of elements in E which are algebraic over F form a subfield of E.