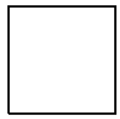
## Math 403 Chapter $3\frac{1}{2}$: Dihedral Groups

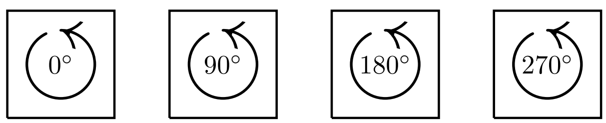
1. **Introduction:** Currently we only have one non-Abelian group, $GL_2\mathbb{R}$, and while we can certainly create more by simply doing $GL_3\mathbb{R}$, $GL_4\mathbb{R}$, and so on, what we'd like to do is find something quite different from these, perhaps even something finite.

2. **The Dihedral Group of order 8:** Imagine we have a square:

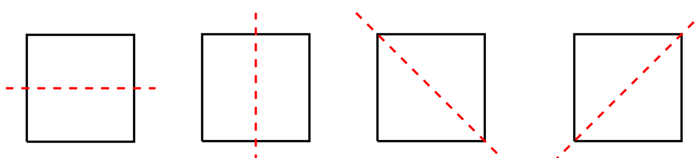


Suppose we may pick this up and put down as long as we put it down on top of the original location. What could we do?

- We could rotate it 0,90,180, or 270 degrees counterclockwise. Call these $R_0$, $R_{90}$, $R_{180}$ and $R_{270}$.



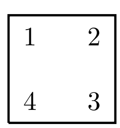

- We could flip it over either a horizontal axis, a vertical axis, or one of two diagonal axes. Call these $H$, $V$, $D$ and $U$ (for Downhill and Uphill axes).



Are there any others? For example what if we flipped and then rotated, would the resulting action be different from one of these eight?

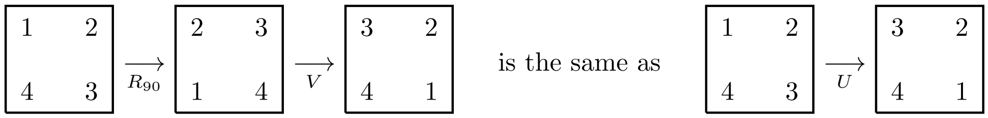
It turns out that the answer is no. Any combination of these actions results in another of these eight actions and these eight actions are all we have.

To help see this consider if we label the corners of the square:




If we pick up and replace the square there are only eight resulting positions. This is because the numbers can either go clockwise or counterclockwise (two choices) and the "1" can be in only one of four positions (four choices).

Now then, to dig deeper, suppose we did $R_{90}$ and then we did $V$. The result is $U$. In other words $VR_{90} = U$.



Note that $VR_{90}$ is read right-to-left. The reason for this is that we are acting on something (the square), which is like a function, and functions are read right-to-left. It might be more accurate to write

$$(V \circ R_{90})\,(\text{square}) = U(\text{square}) \text{ or } V\,(R_{90}\,(\text{square})) = U(\text{square}).$$

If we write down the multiplication table for this group we get the following. In this table each entry contains the product $ab$ where $a$ is taken from the left column and $b$ is taken from the right column. Each entry is then "do $b$ then $a$", or "do row then column".

| | | | | | $b$ | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $ab$ | $R_0$ | $R_{90}$ | $R_{180}$ | $R_{270}$ | $H$ | $V$ | $D$ | $U$ |
| | $R_0$ | $R_0$ | $R_{90}$ | $R_{180}$ | $R_{270}$ | $H$ | $V$ | $D$ | $U$ |
| | $R_{90}$ | $R_{90}$ | $R_{180}$ | $R_{270}$ | $R_0$ | $U$ | $D$ | $H$ | $V$ |
| | $R_{180}$ | $R_{180}$ | $R_{270}$ | $R_0$ | $R_{90}$ | $V$ | $H$ | $U$ | $D$ |
| a | $R_{270}$ | $R_{270}$ | $R_0$ | $R_{90}$ | $R_{180}$ | $D$ | $U$ | $V$ | $H$ |
| | $H$ | $H$ | $D$ | $V$ | $U$ | $R_0$ | $R_{180}$ | $R_{90}$ | $R_{270}$ |
| | $V$ | $V$ | $U$ | $H$ | $D$ | $R_{180}$ | $R_0$ | $R_{270}$ | $R_{90}$ |
| | $D$ | $D$ | $V$ | $U$ | $H$ | $R_{270}$ | $R_{90}$ | $R_0$ | $R_{180}$ |
| | $U$ | $U$ | $H$ | $D$ | $V$ | $R_{90}$ | $R_{270}$ | $R_{180}$ | $R_0$ |

This group is $D_4$, the *dihedral group on a 4-gon*, which has order 8.

Note that this group is non-Abelian, since for example $HR_{90} = D \neq U = R_{90}H$.

3. **The General Dihedral Group:** For any $n \in \mathbb{Z}^+$ we can similarly start with an $n$-gon and then take the group consisting of $n$ rotations and $n$ flips, hence having order $2n$. That is, $D_n$ has $|D_n| = 2n$.

   These are all non-Abelian except for the case $n = 2$.

4. **Interesting Note:** We can in fact define $D_\infty$ if we think of a disk that can be rotated any angle in $[0, 2\pi)$ and flipped over any diameter. We won't go there though.