1. Two of the following sets are well-ordered, and two are not. For the well-ordered ones, prove they are well-ordered. For the other two, give an example of a subset with no least element. The only thing you can assume is that the positive integers are well-ordered.

   (a) The set of positive linear combinations of 25 and 20.
   **Hint:** Yes, because it’s a subset of $\mathbb{Z}^+$.

   (b) The set of rationals between and including 0 and 1.
   **Hint:** No, consider $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$.

   (c) $\{3^n | n \leq -1\}$
   **Hint:** No, the set itself has no least element.

   (d) $\{0.2, 0.22, 0.222, 0.2222, 0.22222, 0.222222, \ldots\}$
   **Hint:** Yes, because if we denote $a_k$ the element $0.\overline{222}\ldots_{\overline{k}}$ then $a_j < a_k$ iff $j < k$ and then it follows from the well-ordering of the positive integers.

2. (a) Use the Prime Number Theorem to find an approximate value for the 1000th prime.
   **Hint:** Use the formula for the approximation of $p_{1000}$.

   (b) Show that $600! + 700! \neq 1300!$.
   **Hint:** Find a divisor of $600! + 700!$ which is not a divisor of $1300!$.

3. Use the Chinese Remainder Theorem to find a solution to the system of congruences:

   \[
   x \equiv 1 \mod 3 \\
   x \equiv 3 \mod 4 \\
   x \equiv 5 \mod 7
   \]

   **Hint:** Straightforward.

4. Find the least nonnegative residue of each of the following. Justify.

   (a) $2^{2^315716} \mod 21$.
   **Hint:** Note that $3 \cdot 7 = 21$ which can help simplify.

   (b) $(-12346)^{589227} \mod 12345$.
   **Hint:** Note that $-12346 \equiv -1 \mod 12345$.

   (c) $2^{257} \mod 19$.
   **Hint:** Find and reduce $2^2, 2^4, 2^8, \ldots, 2^{256}$ and then note that $257 = 256 + 1$.

5. Find all incongruent solutions, if any, to the following:

   (a) $14x \equiv 7 \mod 63$
   **Hint:** Straightforward with the method. One solution can be found by noting $7 \equiv 70$.

   (b) $3x \equiv 1 + 3(2^{162}) \mod 111$
   **Hint:** There are none.

6. Show it is never possible to have $p, q$ and $w = p + q - 2$ all being prime and different from one another. Justify!
   **Hint:** If $p, q$ are both odd then $p, q \geq 3$ and $w = p + q - 2 \geq 4$ and is even, hence not prime. So one of $p, q$ is not odd and hence equals 2. Without loss of generality if $p = 2$ then $w = q$, not distinct.
7. Prove using mathematical induction that \(2^n > (n - 1)^2\) for \(n \geq 4\).

**Hint:** Straightforward induction.

8. Given that \(\gcd(a, 4) = 2\) and \(\gcd(b, 4) = 2\), prove that \(\gcd(a + b, 4) = 4\).

**Hint:** Since \(\gcd(a, 4) = 2\) we know \(2 \mid a\) but \(4 \nmid a\). Thus \(a = 4\alpha + 2\) for some \(\alpha \in \mathbb{Z}\). Likewise \(b = 4\beta + 2\) for some \(\beta \in \mathbb{Z}\). Then \(a + b = 4\alpha + 2 + 4\beta + 2 = 4(\alpha + \beta + 1)\) so that \(4 \mid (a + b)\). Thus \(\gcd(a + b, 4) = 4\) since \(4 \mid 4\) also and nothing larger works.

9. Show that every integer \(n\) can be written as a product \(a^3b\), where \(b\) is not divisible by any cube except 1.

**Hint:** If \(p^k\) is in the PF of \(n\) then put \(p^k = p^{3q}p^r\) where \(0 \leq r < 3\) by the division algorithm. Collect all the \(p^{3q}\) to form \(a^3\) and the remaining will be \(b\).

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**The End**