MATH 406 (JWG) Exam 1 Spring 2021 Solutions

Exam Logistics:

1. From the moment you download this exam you have three hours to take the exam and submit to Gradescope. This includes the entire upload and tag procedure so do not wait until the last minute to do these things.
2. Tag your problems! Please! Pretty please!
3. You may print the exam, write on it, scan and upload.
4. Or you may just write on it on a tablet and upload.
5. Or you are welcome to write the answers on separate pieces of paper if other options don't appeal to you, then scan and upload.

Exam Rules:

1. You may ask for clarification on questions but you may not ask for help on questions!
2. You are permitted to use official class resources which means your own written notes, class Panopto recordings and the textbook.
3. You are not permitted to use other resources. Thus no friends, internet, calculators, Wolfram Alpha, etc.
4. By taking this exam you agree that if you are found in violation of these rules that the minimum penalty will be a grade of 0 on this exam.

Exam Work:

1. Show all work as appropriate for and using techniques learned in this course.
2. Any pictures, work and scribbles which are legible and relevant will be considered for partial credit.
3. Arithmetic calculations do not need to be simplified unless specified.

1. Determine if the following set is well-ordered and justify: [5 pts]

$$\left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z}, b \neq 0,\ a > b \right\}$$

Solution:

Not well ordered. The subset $\left\{ \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, ... \right\}$ has no least element.

2. Simplify: [10 pts]

$$\sum_{j=2}^{n} \left(3^j + \frac{j+1}{2} \right)$$

Solution:

Separately we have:

$$\sum_{j=2}^{n} 3^j = \frac{3^{n+1} - 3}{3 - 1} - 1 - 3$$

And:

$$\frac{1}{2} \sum_{j=2}^{n} j + 1 = \frac{1}{2} \sum_{j=3}^{n+1} j = \frac{1}{2} \left(\left(\frac{(n+1)(n+2)}{2} \right) - 1 - 2 \right)$$

So the result is the sum of these.

3. Use the Prime Number Theorem to find the approximate number of primes between 1000 and 2000. [5 pts]

Solution:

The result is:

$$\frac{2000}{\ln 2000} - \frac{1000}{\ln 1000}$$

4. Suppose $a, b \in \mathbb{Z}$ satisfy $\gcd(a, b) = 2 \cdot 5^2$ and $\operatorname{lcm}(a, b) = 2^2 \cdot 5^2 \cdot 7^2$. What could a and b be? Justify. [5 pts]

Solution:

Since the only primes in the lcm are 2,3,5 we can only have $a = 2^{\alpha_1} 5^{\beta_1} 7^{\gamma_1}$ and $b = 2^{\alpha_2} 5^{\beta_2} 7^{\gamma_2}$.

Since the gcd selects the minimum exponent and the lcm selects the maximum exponent the possibilies are:

- $\min(\alpha_1, \alpha_2) = 1$ and $\max(\alpha_1, \alpha_2) = 2$ so either $\alpha_1 = 1, \alpha_2 = 2$ or the reverse.
- $\min(\beta_1, \beta_2) = 2$ and $\max(\beta_1, \beta_2) = 2$ so $\beta_1 = \beta_2 = 2$.
- $\min(\gamma_1, \gamma_2) = 0$ and $\max(\gamma_1, \gamma_2) = 2$ so either $\gamma_1 = 0, \gamma_2$ or the reverse.

This yields possiblities:

(a) $a = 2^1 5^2 7^0$ and $b = 2^2 5^2 7^2$

(b) $a = 2^2 5^2 7^0$ and $b = 2^1 5^2 7^2$

(c) $a = 2^1 5^2 7^2$ and $b = 2^2 5^2 7^0$

(d) $a = 2^2 5^2 7^2$ and $b = 2^1 5^2 7^0$

Note that (a) and (d) are really the same pair, just reversed, as are (b) and (c).

5. Find all solutions mod $210 = 3 \cdot 5 \cdot 14$ to the following system of congruences: [10 pts]

$$\begin{aligned} x &= 2 \bmod 3 \\ x &= 2 \bmod 5 \\ 2x &= 6 \bmod 14 \end{aligned}$$

Solution: First observe that the third can be simplified to $x = 3$ mod 7 so we'll use that for now.

Let $M = (3)(5)(7) = 105$ and then $m_1 = 35$, $m_2 = 21$ and $m_3 = 15$. We then solve:

- $35y_1 = 1$ mod 3 or $2y_1 = 1$ mod 3 so $y_1 = 2$ mod 3.
- $21y_2 = 1$ mod 5 or $y_2 = 1$ mod 5.
- $15y_3 = 1$ mod 7 or $y_3 = 1$ mod 7.

Thus there is a unique solution mod 105 which is:

$$\begin{aligned} x &= (2)(2)(35) + (2)(1)(21) + (3)(1)(15) \text{ mod } 105 \\ &= 17 \text{ mod } 105 \end{aligned}$$

When we go back to mod 210 we pick up the additional solution $17 + 105 = 122$ thus our two solutions are 17 and 122.

6. Find the least nonnegative residue of each of the following. Justify. [10 pts]

(a) $1! + 2! + 3! + ... + 100!$ modulo 15.

Solution:

Since for $n \geq 5$ we see that $n!$ includes 3 and 5 in the product we have $n! \equiv 0 \bmod 15$. Thus:

$$1! + 2! + 3! + ... + 100! \equiv 1! + 2! + 3! + 4! \equiv 1 + 2 + 6 + 24 \equiv 3 \bmod 15$$

(b) $23471^{589227} \bmod 23472$.

Solution:

Since $23471 \equiv -1 \bmod 23472$ we have:

$$23471^{589227} \equiv (-1)^{589227} \equiv -1 \equiv 23471 \bmod 23472$$

(c) $3^{260} \bmod 15$.

Solution:

Observe that (all mod 15) we have $3^1 \equiv 3$, $3^2 \equiv 9$, $3^4 \equiv 6$, $3^8 \equiv 6$, $3^{16} \equiv 6$, $3^{32} \equiv 6$, $3^{64} \equiv 6$, $3^{128} \equiv 6$, $3^{256} \equiv 6$.

Thus since $260 = 256 + 4$ we have:

$$3^{260} \equiv 3^{256}3^4 \equiv 36 \equiv 6 \bmod 15$$

7. Find all (if any) incongruent solutions modulo the given modulus to the following: [10 pts]

(a) $35x \equiv 10 \mod 55$

Solution:

Since $\gcd(35, 55) = 5 \mid 10$ there are 5 solutions. One of them is $x_0 = 5$ by the Euclidean Algorithm. Thus all solutions have the form:

$$x = 5 + \frac{55}{5}k = 5 + 11k \text{ mod } 55$$

Hence we have:

$$x = 5, 16, 27, 38, 49 \text{ mod } 55$$

(b) $2x \equiv 13^{162} \mod 20$

Solution:

Since $\gcd(2, 20) = 2 \nmid 13^{162}$ there are no solutions.

8. Suppose $n = abc$ is a three-digit number, where each letter is a digit. (For example, 347 has $a = 3$, $b = 4$ and $c = 7$.) Prove that $11 \mid n$ if and only if $11 \mid (a - b + c)$. [10 pts]

Solution:

The quickest way is to observe that:

$$\begin{aligned} 11 \mid n &\iff 100a + 10b + c \equiv 0 \mod 11 \\ &\iff a - b + c + (99a + 11b) \equiv 0 \mod 11 \\ &\iff a - b + c \equiv 0 \mod 11 \\ &\iff 11 \mid a - b + c \end{aligned}$$

9. Use mathematical induction to prove: [10 pts]

$$1(1!) + 2(2!) + ... + n(n!) = (n+1)! - 1 \text{ for all integers } n \geq 1$$

Solution:

For the base case observe that $1(1!) = (1+1)! - 1$.

Assume that for some k we have $1(1!) + 2(2!) + ... + k(k!) = (k+1)! - 1$ and observe that:

$$\begin{aligned}
1(1!) + 2(2!) + ... + (k+1)((k+1)!) &= (k+1)! - 1 + (k+1)(k+1)! \\
&= (k+1)! + (k+1)(k+1)! - 1 \\
&= (k+1)!\,[1 + k + 1] - 1 \\
&= (k+2)! - 1
\end{aligned}$$

10. Prove that if $a, b, c \in \mathbb{Z}$ with $\gcd(a, b) = 1$ and $c \big| (a + b)$ then $\gcd(c, a) = \gcd(c, b) = 1$. [10 pts]

Solution:

First we show that $\gcd(c, a) = 1$. Suppose that $d > 0$ and $d|c$ and $d|a$. Since $d|c$ and $c|(a + b)$ we have $d|(a + b)$. Since $d|(a + b)$ and $d|a$ we have $d|b$. Since $d|a$ and $d|b$ and $\gcd(a, b) = 1$ we have $d = 1$.

A similar proof shows $\gcd(c, b) = 1$.

11. Prove that a perfect square cannot have exactly four distinct positive divisors. [5 pts]

Solution:

Suppose n is a perfect square. Either n is the square of a prime or a composite.

If $n = p^2$ for a prime p then n has three divisors 1, p and n.

If $n = (ab)^2$ then n has at least five divisors 1, a, a^2, a^2b and n.

12. Prove using the Uniqueness of Prime Factorizations that $\sqrt{6}$ is irrational. [10 pts]

Solution:

Suppose $\sqrt{6} = \frac{a}{b}$ for $a, b \in \mathbb{Z}^+$. Then we have:

$$\begin{aligned} a &= b\sqrt{6} \\ a^2 &= 6b^2 \end{aligned}$$

If the prime factorization of a is $a = 2^{\alpha_1}3^{\beta_1}A$ and the prime factorization of b is $b = 2^{\alpha_2}3^{\beta_2}B$ then this gives us:

$$\begin{aligned} \left(2^{\alpha_1}3^{\beta_1}A\right)^2 &= 6\left(2^{\alpha_2}3^{\beta_2}B\right)^2 \\ 2^{2\alpha_1}3^{2\beta_1}A &= 2^{2\alpha_2+1}3^{2\beta_2+1}B \end{aligned}$$

This is a contradiction since the left side has an even number of 2s and 3s but the right side has an odd number.