Math 406 Exam 1 Solutions **Justin Wyss-Gallifent**

Directions: Each numbered question is worth ten points. Coherent, logical justification is required for all problems except those in which counterexamples are requested, those can just be given, or for simple computations.

Note: I've ordered these by difficulty as I perceive it. Your opinion on difficulty might vary, but knowing how I ordered them might help you decide which to do first and which to do last!

1. (a) Find $\pi(18)$.
   Solution: Since $\{2, 3, 5, 7, 11, 13, 17\}$ are all prime we have $\pi(18) = 7$.

   (b) Find the prime factorization of 315.
   Solution: We have $315 = 3^2 \cdot 5 \cdot 7$.

   (c) Show that the set $\left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}^+, a > b \right\}$ is not well-ordered.
   Solution: For example the subset $\left\{ \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, ... \right\}$ has no least element.

   (d) Find how many primes there are, approximately, between one billion and two billion.
   Solution: There are approximately $\frac{2000000000}{\ln(2000000000)} - \frac{1000000000}{\ln(1000000000)} \approx 45131377$ primes.

2. Use the Euclidean Algorithm to find $\gcd(97, 20)$ and then write that gcd as a linear combination of 97 and 20.

   Solution: We have

$$
\begin{aligned}
97 &= (4)20 + (17) \\
20 &= (1)17 + (3) \\
17 &= (5)3 + (2) \\
3 &= (1)2 + (1)
\end{aligned}
$$

   So that $\gcd(97, 20) = 1$. Then:

$$
\begin{aligned}
1 &= 3 - (1)2 \\
&= 3 - (1)(17 - (5)3) \\
&= (6)3 - (1)17 \\
&= (6)(20 - (1)17) - (1)(97 - (4)20) \\
&= (10)20 - (6)17 - (1)97 \\
&= (10)20 - (6)(97 - (4)20) - (1)97 \\
&= (34)20 - (7)97
\end{aligned}
$$

3. Find the number of zeros at the end of 1000! with justification.

   Solution: Consider the PF of 1000!. Zeros at the end are created by multiples of 10 which are pairs of 2's and 5's in the PF so the question is how many such pairs are there.

   Consider that of the numbers 1,2,3,...,1000 we have:
   500 of them are divisible by 2, hence contribute a 2 to the PF.
   250 of them are divisible by 4, contributing another 2 to the PF.
   125 are divisible by 8, contributing another 2 to the PF.
   62 are divisible by 16, contributing another 2.
   31 are divisible by 32, contributing another 2.
   15 are divisible by 64, contributing another 2.
   7 are divisible by 128, contributing another 2.
   3 are divisible by 256, contributing another 2.
   1 is divisible by 512, contributing another 2.

   Thus 1000! has a $2^{500+250+125+62+31+15+7+3+1} = 2^{994}$ in its PF.

   A similar argument, mutatis mutandi with 5's shows that 1000! has 5^{249} in its PF.

   Thus there are 249 pairs of 2's and 5's, hence 249 multiples of 10, hence 249 zeros.

4. The following are all false. Provide counterexamples.

   (a) $a \mid bc$ implies $a \mid b$ or $a \mid c$.
   Solution: For example $6 \mid (3)(4)$ but $6 \nmid 3$ and $6 \nmid 4$.

   (b) $a \mid b$ and $a \mid c$ implies $b \mid c$.
   Solution: For example $2 \mid 4$ and $2 \mid 6$ but $4 \nmid 6$.

   (c) $3 \mid a$ and $3 \mid b$ implies $\gcd(a, b) = 3$.
   Solution: For example $3 \mid 6$ and $3 \mid 12$ but $\gcd(6, 12) = 6 \neq 3$.

5. Simplify $\prod_{j=1}^{n} \left(1 + \frac{2}{j}\right)$. Your result should not have a $\prod$ in it, or any sort of long product.

   Solution: We have

$$\prod_{j=1}^{n} \left(1 + \frac{2}{j}\right) = \prod_{j=1}^{n} \left(\frac{j+2}{j}\right) = \left(\frac{3}{1}\right)\left(\frac{4}{2}\right)\left(\frac{5}{3}\right)\cdots\left(\frac{n+2}{n}\right) = \frac{(n+2)(n+1)}{2}$$

6. Use Mathematical Induction to prove $2^1 + 2^2 + ... + 2^n = 2^{n+1} - 2$ for all integers $n \geq 1$.

   Solution: For the base case $n = 1$ observe that $2^1 = 2^{1+1} - 2$ is true.

   Now assume that

$$2^1 + 2^2 + ... + 2^n = 2^{n+1} - 2$$

   and observe that

$$2^1 + 2^2 + ... + 2^n + 2^{n+1} = 2^{n+1} - 2 + 2^{n+1} = 2 \cdot 2^{n+1} - 2 = 2^{(n+1)+1} - 2$$

   as desired.

7. Find all $n \in \mathbb{Z}$ with $n^2 - 5n + 6$ prime.

   Solution: Observe that $n^2 - 5n + 6 = (n-2)(n-3)$. For this to be prime, one of these must be 1 or -1 and the other must be prime or negative prime. Thus:
   If $n - 2 = 1$ then $n = 3$ then $n^2 - 5n + 6 = 0$ which is not prime.
   If $n - 2 = -1$ then $n = 1$ then $n^2 - 5n + 6 = 2$ which is prime.
   If $n - 3 = 1$ then $n = 4$ then $n^2 - 5n + 6 = 2$ which is prime.
   If $n - 3 = -1$ then $n = 2$ then $n^2 - 5n + 6 = 0$ which is not prime.

   So the only possible values are $n = 1, 4$.

8. Suppose p is a prime and a is a positive integer less than p. Find all possibilities for $\gcd(a, 7a+p)$.

   Solution:
$$\gcd(a, 7a+p) = \gcd(a, 7a+p-7a) = \gcd(a, p)$$
   But since $a < p$ and the only positive divisors of p are 1 and p we don't have $a \mid p$ and so $\gcd(a, p) = 1$.

9. Suppose $a, b \in \mathbb{Z}^+$. Prove that any common divisor of a and b must divide $\gcd(a, b)$.

   Solution: Suppose k divides both a and b. Then k divides all linear combinations of a and b. But the gcd is a linear combination of a and b so that k divides the gcd.

10. Prove that for $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ that if $a^n \mid b^n$ then $a \mid b$.
    Note: Tricky.

    Solution: Suppose that $a^n \mid b^n$. Then $ka^n = b^n$ for some $k \in \mathbb{Z}$.

    For any prime that appears in the prime factorization of k, that prime must appear with a power which is a multiple of n, since it appears in b^n with a power which is a multiple of n and if it appears in a^n it must also be with a power which is a multiple of n.

    But this means $k = p_1^{c_1 n} ... p_m^{c_m n}$ is the prime factorization of k and so $k = (p_1^{c_1} ... p_m^{c_m})^n$ is a perfect square, meaning $\sqrt{k} \in \mathbb{Z}^+$, so that $a\sqrt{k} = b$ and $a \mid b$.

The End $\ddot\smile$