MATH 406 Exam 1 Summer 2021

Due by Saturday July 24 at 1:00pm

Exam Logistics:

1. From the moment you download this exam you have three hours to take the exam and submit to Gradescope. This includes the entire upload and tag procedure so do not wait until the last minute to do these things.
2. Tag your problems! Please! Pretty please!
3. You may print the exam, write on it, scan and upload.
4. Or you may just write on it on a tablet and upload.
5. Or you are welcome to write the answers on separate pieces of paper if other options don't appeal to you, then scan and upload.

Exam Rules:

1. You may ask for clarification on questions but you may not ask for help on questions!
2. You are permitted to use official class resources which means your own written notes, class recordings and the textbook.
3. You are not permitted to use other resources. Thus no friends, internet, calculators, Wolfram Alpha, etc.
4. By taking this exam you agree that if you are found in violation of these rules that the minimum penalty will be a grade of 0 on this exam.

Exam Work:

1. Show all work as appropriate for and using techniques learned in this course.
2. Any pictures, work and scribbles which are legible and relevant will be considered for partial credit.
3. Arithmetic calculations do not need to be simplified unless specified.

1. For each of the following: If a set is well-ordered, no justification is required. If a set is not well-ordered, justification is required. If a set is neither, explain why this is the case. [5 pts]

   (a) $A = \left\{ \frac{a}{b} \middle| a, b \in \mathbb{Z}^+ \text{ and } a > b \right\}$

   Solution:

   No, the subset $\left\{ \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, ... \right\}$ has no least element.

   (b) $B = \mathbb{Q} \cap \left[\frac{1}{2}, \infty \right)$

   Solution:

   No, the subset of rationals greater than $\frac{1}{2}$ has no least element.

   (c) $C = \{1, 2\}$

   Solution:

   Yes.

   (d) $D = \mathbb{R} \times \mathbb{R}$

   Solution:

   Neither since "less than" makes no sense here.

2. Use the Euclidean Algorithm to calculate $\gcd(816, 84)$ and to write the gcd as a linear combination of the two. [10 pts]

Solution:

We have:

$$\begin{aligned} 816 &= 9(84) + 60 \\ 84 &= 1(60) + 24 \\ 60 &= 2(24) + 12 \\ 24 &= 2(12) + 0 \end{aligned}$$

Thus $\gcd(816, 84) = 12$.

Then observe that:

$$\begin{aligned} 12 &= 60 - 2(24) \\ 12 &= 60 - 2(84 - 1(60)) \\ 12 &= 3(60) - 2(84) \\ 12 &= 3(816 - 9(84)) - 2(84) \\ 12 &= 3(816) - 29(84) \end{aligned}$$

3. The following can be evaluating using some sort of simplification rather than brute calculation. [10 pts] Do so, showing the steps.

(a) The sum:

$$\sum_{k=4}^{1000}(-1)^k\left(\frac{1}{k}+\frac{1}{k+1}\right)$$

Solution:
This sum telescopes:

$$\begin{aligned}\sum_{k=4}^{1000}(-1)^k\left(\frac{1}{k}+\frac{1}{k+1}\right) &= \left(\frac{1}{4}+\frac{1}{5}\right)-\left(\frac{1}{5}+\frac{1}{6}\right)+...-\left(\frac{1}{999}+\frac{1}{1000}\right)+\left(\frac{1}{1000}+\frac{1}{1001}\right)\\ &= \frac{1}{4}+\frac{1}{1001}\end{aligned}$$

(b) The sum:

$$\sum_{i=1}^{20}(-1)^i\frac{2^i}{3^{2i+1}}$$

Solution:
This sum is geometric:

$$\begin{aligned}\sum_{i=1}^{20}(-1)^i\frac{2^i}{3^{2i+1}} &= \sum_{i=1}^{20}(-1)^i\frac{2^i}{(3^2)^i 3}\\ &= \frac{1}{3}\sum_{i=1}^{20}(-1)^i\frac{2^i}{9^i}\\ &= \frac{1}{3}\sum_{i=1}^{20}\left(-\frac{2}{9}\right)^i\\ &= \frac{1}{3}\left(\frac{(-2/9)^{21}-1}{(-2/9)-1}\right)\end{aligned}$$

4. The following can be evaluating using some sort of simplification rather than brute calculation. [5 pts] Do so, showing the steps.

The product:

$$\prod_{k=3}^{123}\left(1-\frac{1}{k^2}\right)$$

Solution:

Observe that:

$$1-\frac{1}{k^2}=\frac{k^2-1}{k^2}=\frac{(k-1)(k+1)}{k^2}$$

We can rewrite the product:

$$\prod_{k=3}^{123}\left(1-\frac{1}{k^2}\right)=\prod_{k=3}^{123}\frac{(k-1)(k+1)}{k^2}=\prod_{k=3}^{123}\frac{k-1}{k}\prod_{k=3}^{123}\frac{k+1}{k}$$

Each telescopes:

$$\begin{aligned}\prod_{k=3}^{123}\frac{k-1}{k}\prod_{k=3}^{123}\frac{k+1}{k}&=\left[\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{121}{122}\right)\left(\frac{122}{123}\right)\right]\left[\left(\frac{4}{3}\right)\left(\frac{5}{4}\right)\cdots\left(\frac{123}{122}\right)\left(\frac{124}{123}\right)\right]\\&=\frac{2}{123}\cdot\frac{124}{3}\end{aligned}$$

5. Which pairs of positive integers a and b could have $\gcd(a,b) = 20$ and $\text{lcm}(a,b) = 400$. Justify systematically. [10 pts]

Solution:

Consider that $\gcd(a,b) = 2^2 \cdot 5^1$ and $\text{lcm}(a,b) = 2^4 \cdot 5^2$.

This means that we know that in a and b:

- The smallest power of 2 is 2 and the largest power of 2 is 4.
- The smallest power of 5 is 1 and the largest power of 5 is 2.

We then have the following possibilities:

- $a = 2^2 \cdot 5^1 = 20$ and $b = 2^4 \cdot 5^2 = 400$
- $a = 2^4 \cdot 5^1 = 80$ and $b = 2^2 \cdot 5^2 = 100$
- $a = 2^2 \cdot 5^2 = 100$ and $b = 2^4 \cdot 5^1 = 80$
- $a = 2^4 \cdot 5^1 = 400$ and $b = 2^2 \cdot 5^2 = 20$

6. Use Well-Ordering to prove that for all $n \in \mathbb{N}$ that: [10 pts]

$$1 + 5 + 5^2 + ... + 5^n = \frac{5^{n+1} - 1}{4}$$

Solution:

Suppose not. Let S be the set of naturals for which the statement is false and let k be the smallest natural for which the statement is false.

Clearly $k \neq 1$ since $1 + 5 = \frac{5^2-1}{4}$ so the statement is true for $k = 1$.

We know the statement is true for $k - 1$ since $k - 1 < k$, so we know that:

$$1 + 5 + ... + 5^{k-1} = \frac{5^k - 1}{4}$$

From here:

$$\begin{aligned}
1 + 5 + ... + 5^{k-1} &= \frac{5^k - 1}{4} \\
1 + 5 + ... + 5^{k-1} + 5^k &= \frac{5^k - 1}{4} + 5^k \\
1 + 5 + ... + 5^{k-1} + 5^k &= \frac{5^k - 1}{4} + \frac{4 \cdot 5^k}{4} \\
1 + 5 + ... + 5^{k-1} + 5^k &= \frac{5 \cdot 5^k - 1}{4} \\
1 + 5 + ... + 5^{k-1} + 5^k &= \frac{5^{k+1} - 1}{4}
\end{aligned}$$

This contradicts the fact that the statement is false for k.

7. Use Weak Induction to prove that for all $n \in \mathbb{N}$ that $4^n > n^2$. [10 pts]

Solution:

For the base case we check if $4^1 \geq 1^2$ which is true.

For the inductive step we assume that $4^k > k^2$ and we claim that $4^{k+1} > (k+1)^2$. Observe that:

$$4^{k+1} = 4 \cdot 4^k > 4k^2$$

So we need to ensure that $4k^2 \geq (k+1)^2$. To see this observe that:

$$4k^2 - (k+1)^2 = 4k^2 - k^2 - 2k - 1 = 3k^2 - 2k - 1 = k(3k-2) - 1$$

Since $k \geq 1$ we know that $k(3k-2) - 1 \geq 1(3(1) - 2) - 1 = 0$ and we are done.

8. Define a sequence recursively by: [10 pts]

$$a_0 = 1 \text{ and } a_1 = 3 \text{ and } a_n = 2a_{n-1} - a_{n-2} \text{ for } n \geq 2$$

Use Strong Induction to prove that for all $n \geq 0$ that $a_n = 2n + 1$.

Solution:

For the inductive step we assume all of:

$$\begin{aligned} a_0 &= 2(0) + 1 \\ a_1 &= 2(1) + 1 \\ \vdots &= \vdots \\ a_{k-1} &= 2(k-1) + 1 \\ a_k &= 2k + 1 \end{aligned}$$

We claim $a_{k+1} = 2(k+1) + 1$. Observe that:

$$a_{k+1} = 2a_k - a_{k-1} = 2(2k+1) - (2(k-1)+1) = 4k + 2 - 2k + 1 = 2k + 3$$

This is what we want.

For the base cases observe that we reference $k-1$ in the inductive step and so we need $k-1 \geq 0$ or $k \geq 1$. Thus we must do $n = 0$ and $n = 1$ as base cases.

- For $n = 0$ we check if $a_0 = 2(0) + 1$ which is $1 = 1$ which is true.
- For $n = 1$ we check if $a_1 = 2(1) + 1$ which is $3 = 3$ which is true.

9. Let S be the set of positive real numbers which have a finite decimal expansion (meanining you can write them as a decimal which doesn't go on forever). Show that S is countable by clearly exhibiting the construction of a one-to-one mapping with $\mathbb{N}$. [15 pts]

Solution:

Note: There are several ways to approach this, so please keep in mind that this is just the solution I came up with first. Several students had very different but equally valid solutions.

Onwards - Every such real number is equal to some positive integer divided by a power of 10. So first we construct a grid that contains all of these.

To do this, create a grid where the entry in row i and column j equals the integer j divided by 10^{i-1}.

Thus the first row consists of all positive integers. The second row consists of all positive integers divided by 10. The third row consists of all positive integers divided by 100. And so on.

$$
\begin{array}{llllllllllll}
1 & 2 & 3 & ... & 9 & 10 & 11 & ... & 99 & 100 & 101 & ... \\
0.1 & 0.2 & 0.3 & ... & 0.9 & 1.0 & 1.1 & ... & 9.9 & 10.0 & 10.1 & ... \\
0.01 & 0.02 & 0.03 & ... & 0.09 & 0.10 & 0.11 & ... & 0.99 & 1.00 & 1.01 & ... \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Then zig-zag through like in the proof from class, skipping repeats.

10. Prove that: [15 pts]

$$a \mid bc \text{ iff } \left.\frac{a}{\gcd(a,b)}\right| c$$

Solution:

Note: There are several ways to approach this, so please keep in mind that this is just the solution I came up with first. Several students had very different but equally valid solutions.

$\Longrightarrow$

If $a \mid bc$ then there is some $x \in \mathbb{Z}$ such that $ax = bc$. Dividing through by $\gcd(a,b)$ yields:

$$\frac{a}{\gcd(a,b)}x = \frac{b}{\gcd(a,b)}c$$

This tells us that:

$$\frac{a}{\gcd(a,b)} \mid \frac{b}{\gcd(a,b)}c$$

And since the two fractions are coprime (theorem from class) we get:

$$\left.\frac{a}{\gcd(a,b)}\right| c$$

As desired.

$\Longleftarrow$

On the other hand suppose that:

$$\left.\frac{a}{\gcd(a,b)}\right| c$$

Then there is some $x \in \mathbb{Z}$ such that:

$$\frac{a}{\gcd(a,b)}x = c$$

Rewriting this as $ax = \gcd(a,b)c$ tells us that $a \mid \gcd(a,b)c$ and since $\gcd(a,b) \mid b$ we know $\gcd(a,b)c \mid bc$ and so $a \mid bc$.