MATH 406 (JWG) Exam 2 Spring 2021

Due by Friday 9 at 12:00pm

Exam Logistics:

1. From the moment you download this exam you have three hours to take the exam and submit to Gradescope. This includes the entire upload and tag procedure so do not wait until the last minute to do these things.
2. Tag your problems! Please! Pretty please!
3. You may print the exam, write on it, scan and upload.
4. Or you may just write on it on a tablet and upload.
5. Or you are welcome to write the answers on separate pieces of paper if other options don't appeal to you, then scan and upload.

Exam Rules:

1. You may ask for clarification on questions but you may not ask for help on questions!
2. You are permitted to use official class resources which means your own written notes, class Panopto recordings and the textbook.
3. You are not permitted to use other resources. Thus no friends, internet, calculators, Wolfram Alpha, etc.
4. By taking this exam you agree that if you are found in violation of these rules that the minimum penalty will be a grade of 0 on this exam.

Exam Work:

1. Show all work as appropriate for and using techniques learned in this course.
2. Any pictures, work and scribbles which are legible and relevant will be considered for partial credit.
3. Arithmetic calculations do not need to be simplified unless specified.

1. Identify which of $0, 1, ..., 13$ are coprime to 14 and for each which is coprime determine the order of each mod 14. Then state which are primitive roots. [6 pts]

   Solution:

2. Calculate $\phi(\sigma(\tau(1000)))$ and simplify. [5 pts]

Solution:

3. Find all numbers n satisfying both $\phi(n) = 20$ and $\tau(n) \mid 4$. [10 pts]

Solution:

4. It's a fact that $r = 2$ is a primitive root mod 13.

   (a) Use this to construct a table of indices for this primitive root. [5 pts]

   Solution:

   (b) Use the table of indices to solve the equation: $4^x \equiv 12 \mod 13$. Your answer(s) should be mod 12. [5 pts]

   Solution:

(c) Use the table of indices to solve the equation: $x^2 \equiv 12 \mod 13$. Your answer(s) should be mod 13. [5 pts]

Solution:

(d) Find the least nonnegative residues of all of the other primitive roots mod 13. [5 pts]

Solution:

5. Suppose $a, n \in \mathbb{Z}^+$. Let $m = a^n - 1$. Prove that $\text{ord}_m a = n$. [10 pts]

Solution:

6. Prove that if $a, m \in \mathbb{Z}^+$ with $\gcd(a, m) = \gcd(a-1, m) = 1$ then: [10 pts]

$$1 + a + a^2 + ... + a^{\phi(m)-1} \equiv 0 \mod m$$

Solution:

7. Show that a positive integer n is composite iff $\phi(n) \leq n - \sqrt{n}$. [12 pts]

Solution:

8. Define $f(n) = \gcd(n, 6)$. Prove that $f(n)$ is multiplicative. [10 pts]

Solution:

9. Let p be an odd prime and t a positive integer. Show that p^t and $2p^t$ have the same number of primitive roots. You can assume that they both do actually have primitive roots. [5 pts]

Solution:

10. Let n be a positive integer. Prove that the product of the divisors of n equals $n^{\tau(n)/2}$. [12 pts]

Hint: The case where n is not a perfect square is easier so do it first.

Solution: