Math 406 Exam 2 **Justin Wyss-Gallifent**

Directions: Each numbered question is worth ten points. Coherent, logical justification (including words and clear calculations) is required for all problems except those in which counterexamples are requested, those can just be given, or for simple computations.

Note: I've ordered these by difficulty as I perceive it. Your opinion on difficulty might vary, but knowing how I ordered them might help you decide which to do first and which to do last!

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1. Use the CRT to find the second smallest positive integer solution to the following system:

$$\begin{aligned} 3x &\equiv 6 \bmod 15 \\ 5x &\equiv 4 \bmod 6 \\ x+1 &\equiv 2 \bmod 7 \end{aligned}$$

2. Find each of the following.
   (a) The least nonnegative residue of $(14!)4^{371}$ modulo 17.
   (b) The least nonnegative residue of 1234^5 modulo 1236.

3. Find all incongruent solutions, if any, modulo the original modulus, to the following:
   (a) $5x \equiv 6 \bmod 16$
   (b) $2x \equiv 18 \bmod 46$
   (c) $13^{162}x \equiv 2 \bmod 13^{163}$

4. Calculate the following. Answers do not need to be simplified!
   (a) $\phi(6!7!)$
   (b) $\sigma(10^{10})$
   (c) $\tau(10!)$

5. Show that 91 is a Fermat Pseudoprime to the base 3. Note that 91 is not prime!

6. Prove that if $n \geq 2$ and $\gcd(6, n) = 1$ then $\phi(3n) = 2\phi(2n)$.

7. Classify all numbers n for which $\tau(n) = 12$.

8. Prove (using the definition of congruence) or disprove (by counterexample) each of the following. Hint: One is true, two are false.
   (a) If $ac \equiv bc \bmod m$ with $c \not\equiv 0 \bmod m$ then $a \equiv b \bmod m$.
   (b) If $a \equiv b \bmod m$ and $b \equiv c \bmod m$ then $a \equiv c \bmod m$.
   (c) If $a \equiv b \bmod m$ then $m \mid (a + b)$.

9. Suppose n is a perfect number and p is a prime such that pn is also perfect. Prove $\gcd(p, n) \neq 1$.

10. Prove that for a fixed k that $\phi(n) = k$ can have at most a finite number of solutions.

Beginning $\equiv$ End mod (Length of Exam)