MATH 406: Homework 1

1. Determine whether each of the following sets is well-ordered. If so, give a proof which relies on the fact that \( \mathbb{Z}^+ \) is well-ordered. If not, give an example of a subset with no least element.

   (a) \( \{ a \mid a \in \mathbb{Z}, a > 3 \} \)
   (b) \( \{ a \mid a \in \mathbb{Q}, a > 3 \} \)
   (c) \( \{ \frac{a}{2} \mid a \in \mathbb{Z}, a \geq 10 \} \)
   (d) \( \{ \frac{2a}{3} \mid a \in \mathbb{Z}, a > 10 \} \)

2. Suppose \( a, b \in \mathbb{Z}^+ \) are unknown. Let \( S = \{ a - bk \mid k \in \mathbb{Z}, a - bk > 0 \} \). Explain why \( S \) has a smallest element but no largest element.

3. Use the well-ordering property to show that \( \sqrt{5} \) is irrational.

4. Use the identity
   \[
   \frac{1}{k^2 - 1} = \frac{1}{2} \left( \frac{1}{k - 1} - \frac{1}{k + 1} \right)
   \]
   to evaluate the following:

   (a) \( \sum_{k=2}^{10} \frac{1}{k^2 - 1} \)
   (b) \( \sum_{k=2}^{n} \frac{1}{k^2 - 1} \)
   (c) \( \sum_{k=1}^{n} \frac{1}{k^2 + 2k} \) Hint: \( k^2 + 2k = (???)^2 - 1 \)

5. Find the value of each of the following:

   (a) \( \prod_{j=2}^{7} \left( 1 - \frac{1}{j} \right) \)
   (b) \( \prod_{j=2}^{n} \left( 1 - \frac{1}{j} \right) \)
   (c) \( \prod_{j=2}^{n} \left( 1 - \frac{1}{j^2} \right) \) Hint: Be sneaky!

6. Use weak mathematical induction to prove that
   \[
   \sum_{j=1}^{n} j(j + 1) = \frac{n(n + 1)(n + 2)}{3}
   \]
   for every positive integer \( n \).

7. Use Weak Mathematical Induction to show that \( f_n f_{n+2} = f_{n+1}^2 + (-1)^{n+1} \) for all \( n \geq 1 \).

8. Use weak mathematical induction to show that a \( 2^n \times 2^n \) chessboard with a corner missing
   can be tiled with pieces shaped like \( \begin{array}{c} \square \end{array} \) for every integer \( n \geq 0 \).

9. Define:
   \[
   H_n = \sum_{j=1}^{n} \frac{1}{j}
   \]
   Use weak mathematical induction to prove that for all \( n \geq 1 \) we have \( H_{2^n} \leq 1 + n \).

10. Use strong mathematical induction to prove that every amount of postage over 53 cents can be formed using 7-cent and 10-cent stamps.