1. Find the number of positive integers $\leq 1000$ that are not divisible by 3 or 5. Do not do this by brute force. Explain your method.

**Solution:** The number of integers less than or equal to $n$ which are divisible by $d$ is $\left\lfloor \frac{n}{d} \right\rfloor$, where the symbol means the greatest integer less than or equal to $n/d$.

Therefore the number of positive integers $\leq 1000$ which are divisible by 3 is $\left\lfloor \frac{1000}{3} \right\rfloor = 333$ so we throw those out, leaving $1000 - 333 = 667$, the number which are divisible by 5 is $\left\lfloor \frac{1000}{5} \right\rfloor = 200$ so we throw those out, leaving $667 - 200 = 467$. However there are $\left\lfloor \frac{1000}{15} \right\rfloor = 66$ integers which are divisible by 15 were thrown out twice so we must add them back in, giving $467 + 66 = 533$.

2. Suppose $\gcd(b,c) = 3$ and $\gcd(a,b) = 2$. Prove $\gcd(c,a+2b) \neq c$.

**Proof:** Assume that $\gcd(c,a+2b) = c$. Now then, since $\gcd(b,c) = 3$ we have $3|b$ and $3|c$. Together $3|c$ and $3|(a+2b)$ tells us $3|(a+2b)$ but then $3|a$ because $a = (a+2b) - 2(b)$, a linear combination of $a+2b$ and $b$. So now $3|a$ and $3|b$ but $\gcd(a,b) = 2$, a contradiction.

3. Find each of the following however you want. You don’t need to show work. Wolfram Alpha can do it!

(a) $\pi(10)$

**Solution:** $\pi(10) = 4$

(b) $\pi(100)$

**Solution:** $\pi(100) = 25$

(c) $\pi(1000)$

**Solution:** $\pi(1000) = 168$

(d) $\pi(10000)$

**Solution:** $\pi(10000) = 1229$

4. Show that there is always a prime greater than $n \in \mathbb{Z}^+$ by exploring the number $n! + 1$.

**Solution:** We know that every positive integer greater than 2 has a prime divisor. Since $n! + 1$ is not divisible by $2, ..., n$ (because $k | n! + 1$ and $2 \leq k \leq n$ implies $k | 1$, a contradiction) it has a prime divisor which is greater than $n$. Thus there is always a prime greater than any given $n$ and so there are infinitely many primes.

5. Using the theorem from class find 10 consecutive composite integers. Then find (however you can) the smallest 10 consecutive composite integers.

**Solution:** By the Theorem from class we have the following:

$(10 + 1)! + 2 = 39916803$

$(10 + 1)! + 3 = 39916804$

$(10 + 1)! + 4 = 39916805$

$(10 + 1)! + 5 = 39916806$

$(10 + 1)! + 6 = 39916807$

$(10 + 1)! + 7 = 39916808$

$(10 + 1)! + 8 = 39916809$

$(10 + 1)! + 9 = 39916810$

$(10 + 1)! + 10 = 39916811$

$(10 + 1)! + 11 = 39916812$

The smallest 10 consecutive composite integers are $114,115,116,117,118,119,120,121,122,123$. 
6. Find all $n \in \mathbb{Z}$ such that $n^3 + 1$ is prime.  
**Hint:** $n^3 + 1$ factors. What must one of the factors be?  
**Solution:** Observe that $n^3 + 1 = (n + 1)(n^2 - n + 1)$ so for $n^3 + 1$ to be prime we must have either $n + 1 = 1$ or $n^2 - n + 1 = 1$. The former gives $n = 0$ but then $n^3 + 1 = 1$ is not prime. The latter gives $n^2 - n = 0$ so that $n = 0$ or $n = 1$. Again $n = 0$ is no good so $n = 1$ and we see $n^3 + 1 = 2$ is prime.

7. For each of $n = 10, 100, 1000, 1M, 1B$ find $\pi(n)$ exactly and approximate it via the PNT. Calculate the percentage error $100|\text{exact} - \text{approx}|/\text{exact}$ for each.  
**Solution:** We have the following:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\pi(n)$</th>
<th>$n/\ln(n) \approx$</th>
<th>% Error $\approx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4</td>
<td>4.34</td>
<td>8.57</td>
</tr>
<tr>
<td>100</td>
<td>25</td>
<td>21.71</td>
<td>13.14</td>
</tr>
<tr>
<td>1000</td>
<td>168</td>
<td>144.76</td>
<td>13.83</td>
</tr>
<tr>
<td>1M</td>
<td>78498</td>
<td>72382.41</td>
<td>7.79</td>
</tr>
<tr>
<td>1B</td>
<td>50847534</td>
<td>48254942.43</td>
<td>5.10</td>
</tr>
</tbody>
</table>

8. For each of $n = 10, 100, 1000, 1M, 1B$ find the $n^{th}$ prime exactly and approximate it via the PNT Corollary. Calculate the percentage error $100|\text{exact} - \text{approx}|/\text{exact}$ for each.  
**Solution:** We have the following:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$p_n$</th>
<th>$n \ln(n) \approx$</th>
<th>% Error $\approx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>29</td>
<td>23.02</td>
<td>20.60</td>
</tr>
<tr>
<td>100</td>
<td>541</td>
<td>460.52</td>
<td>14.88</td>
</tr>
<tr>
<td>1000</td>
<td>7919</td>
<td>6907.76</td>
<td>12.77</td>
</tr>
<tr>
<td>1M</td>
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<td>13815510.56</td>
<td>10.79</td>
</tr>
<tr>
<td>1B</td>
<td>22801763489</td>
<td>20723265836.95</td>
<td>9.12</td>
</tr>
</tbody>
</table>

9. Find a prime between $n$ and $2n$ for:  
(a) $n = 10$  
**Solution:** 17 is prime.  
(b) $n = 711$  
**Solution:** 719 is prime.  
(c) $n = 31415$  
**Solution:** 39989 is prime.  

10. Verify the Goldbach Conjecture for:  
(a) $n = 100$  
**Solution:** $100 = 3 + 97$  
(b) $n = 462$ **Solution:** $462 = 5 + 457$  
(c) $n = 4538674$  
**Solution:** $4538674 = 3 + 4538671$
11. Let $S = \{1\} \cup \{\text{primes}\}$. Prove that every positive integer can be written as a sum of distinct elements of $S$.

Hint: This is challenging, don’t be afraid to ask for a hint!

**Proof:**

We proceed by strong induction on $n \in \mathbb{Z}^+$.

**Base Case:** $n = 1$ may be written as the sum $1 = 1$.

**Inductive Step:** Assume that all of $1, 2, ..., n - 1$ may be written as the sum of distinct elements of $S$. Consider $n$.

If $n$ is even use Bertrand’s Postulate to find a prime $p$ satisfying $\frac{n}{2} < p < n$. Then $n = p + (n - p)$. Since $n - p < n$ the inductive hypothesis states that $n - p$ is the sum of distinct elements of $S$, that is $n - p = s_1 + ... + s_k$, and so $n = s_1 + ... + s_k + p$. Notice that $p$ is not the same as any of the $s_i$ because $s_1 + ... + s_k = n - p < n - \frac{n}{2} = \frac{n}{2} < p$.

If $n$ is odd use Bertrand’s Postulate to find a prime $p$ satisfying $\frac{n+1}{2} < p < n + 1$. Then $n = p + (n - p)$. It’s possible that $p = n$ in which case $n - p = 0$ which is fine because $n = p$ and we’re done, otherwise the inductive hypothesis states that $n - p$ is the sum of distinct elements of $S$, that is $n - p = s_1 + ... + s_k$, and so $n = s_1 + ... + s_k + p$. Notice that $p$ is not the same as any of the $s_i$ because $s_1 + ... + s_k = n - p < n - \frac{n+1}{2} = \frac{n-1}{2} < \frac{n+1}{2} < p$. 