MATH 406: Homework 3 Solutions

1. Use the Euclidean Algorithm to calculate \( d = \gcd(510, 140) \) and then use the result to find \( \alpha \) and \( \beta \) so that \( d = 510\alpha + 140\beta \).

Solution:

\[
\begin{align*}
510 &= 3 \cdot 140 + 90 \\
140 &= 1 \cdot 90 + 50 \\
90 &= 1 \cdot 50 + 40 \\
50 &= 1 \cdot 40 + 10 \\
40 &= 4 \cdot 10 + 0
\end{align*}
\]

so that \( \gcd(510, 140) = 10 \). Then observe:

\[
\begin{align*}
10 &= 50 - 1 \cdot 40 \\
10 &= 50 - 1(90 - 1 \cdot 50) \\
10 &= 2 \cdot 50 - 1 \cdot 90 \\
10 &= 2(140 - 1 \cdot 90) - 1 \cdot 90 \\
10 &= 2 \cdot 140 - 3 \cdot 90 \\
10 &= 2 \cdot 140 - 3(510 - 3 \cdot 140) \\
10 &= 11 \cdot 140 - 3 \cdot 510
\end{align*}
\]

2. Use the Euclidean Algorithm to show that if \( k \in \mathbb{Z}^+ \) that \( 3k + 2 \) and \( 5k + 3 \) are relatively prime.

Solution:

\[
\begin{align*}
5k + 3 &= 1(3k + 2) + (2k + 1) \\
3k + 2 &= 1(2k + 1) + (k + 1) \\
2k + 1 &= 1(k + 1) + (k) \\
k + 1 &= 1(k) + 1 \\
k &= k(1) + 0
\end{align*}
\]

3. How many zeros are there at the end of \( (1000)! \)? Do not do this by brute force. Explain your method.

Solution: Consider the PF of 1000!. Zeros at the end are created by multiples of 10 which are pairs of 2’s and 5’s in the PF so the question is how many such pairs are there.

Consider that of the numbers 1, 2, 3, ..., 1000 we have:

- 500 of them are divisible by 2, hence contribute a 2 to the PF.
- 250 of them are divisible by 4, contributing another 2 to the PF.
- 125 are divisible by 8, contributing another 2 to the PF.
- 62 are divisible by 16, contributing another 2.
- 31 are divisible by 32, contributing another 2.
- 15 are divisible by 64, contributing another 2.
- 7 are divisible by 128, contributing another 2.
- 3 are divisible by 256, contributing another 2.
- 1 is divisible by 512, contributing another 2.

Thus 1000! has \( 2^{500+250+125+62+31+15+7+3+1} = 2^{994} \) in its PF.

A similar argument, mutatis mutandi with 5’s shows that 1000! has \( 5^{249} \) in its PF.

Thus there are 249 pairs of 2’s and 5’s, hence 249 multiples of 10, hence 249 zeros.
4. Let \( a = 1038180 \) and \( b = 92950 \). First find the prime factorizations of \( a \) and \( b \). Then use these to calculate \( \gcd(a, b) \) and \( \text{lcm}(a, b) \).

**Solution:** We have \( 1038180 = 2^23^15^111^313^1 \) and \( 92950 = 2^15^111^113^2 \) and so \( \gcd = 2^15^111^113^1 \) and \( \text{lcm} = 2^23^15^211^313^2 \)

5. Which pairs of integers have \( \gcd \) of 18 and \( \text{lcm} \) of 540? Explain. **Solution:** Observe that 18 = \( 2^1 \cdot 3^2 \) and 540 = \( 2^23^35^1 \). Suppose \( a \) and \( b \) have \( \gcd \) of 18 and \( \text{lcm} \) of 540. The only primes in the \( \text{PF} \) of \( a \) and \( b \) are 2, 3 and 5, since if another prime divided one of them it would appear in the \( \text{lcm} \).

Thus \( a = 2^a3^b5^c \) and \( b = 2^d3^e5^f \).

Now then, for the \( \gcd \) we take the minimum power of the common primes, so the minimum of \( a \) and \( \beta \) must be 1, the minimum of \( b \) and \( \beta \) must be 2 and the minimum of \( c \) and \( \gamma \) must be 0.

For the \( \text{lcm} \) we take the maximum power of the common primes, so the maximum of \( a \) and \( \alpha \) must be 2, the maximum of \( b \) and \( \beta \) must be 3 and the maximum of \( c \) and \( \gamma \) must be 1.

Up to interchanging \( a \) and \( b \) we therefore have 4 possibilities:

\[
\begin{align*}
a &= 2^13^50, \quad b = 2^23^35^1 \\
a &= 2^13^51, \quad b = 2^23^35^0 \\
a &= 2^13^50, \quad b = 2^23^25^1 \\
a &= 2^13^51, \quad b = 2^23^25^0
\end{align*}
\]

6. Suppose that \( a \in \mathbb{Z} \) is a perfect square divisible by at least two distinct primes. Show that \( a \) has at least seven distinct factors.

**Solution:** Since \( a \) is a perfect square, let \( a = b^2 \) and let \( b = p_1^{k_1} \ldots p_n^{k_n} \) be the prime factorization of \( b \). Then \( a = b^2 = p_1^{2k_1} \ldots p_n^{2k_n} \). Since \( a \) is divisible by at least two distinct primes we know that \( n \geq 2 \). Then observe that \( 1, p_1, p_2, p_1p_2, p_1^2, p_2^2, a \) are all distinct factors of \( a \).

7. Show that if \( a, b \in \mathbb{Z}^+ \) with \( a^3 \mid b^2 \) then \( a \mid b \).

**Solution:** We show that for any \( p^k \) appearing in the \( \text{PF} \) of \( a \) that \( p^l \) appears in the \( \text{PF} \) of \( b \) with \( l \geq k \).

Suppose \( p^k \) appears in the \( \text{PF} \) of \( a \). Then \( p^{3k} \) appears in the \( \text{PF} \) of \( a^3 \). Since \( a^3 \mid b^2 \) and since the exponents of the prime factors of \( b^2 \) are all even, we know that \( p^{2l} \) appears in the \( \text{PF} \) of \( b^2 \) with \( 2l \geq 3k \). Then we have \( p^l \) appearing in the \( \text{PF} \) of \( b \) and since \( l \geq \frac{3}{2}k > k \) we have our claim.

8. For which positive integers \( m \) is each of the following statements true:

(a) \( 34 \equiv 10 \mod m \)
   **Solution:** We need \( m \mid 34 - 10 \) or \( m \mid 24 \), so we could have \( m = 2, 3, 4, 6, 8, 12 \) or 24.

(b) \( 1000 \equiv 1 \mod m \)
   **Solution:** We need \( m \mid 1000 - 1 \) or \( m \mid 999 \), so we could have \( m = 3, 9, 27, 37, 999, 111 \) or 333.

(c) \( 100 \equiv 0 \mod m \)
   **Solution:** We need \( m \mid 100 - 0 \) or \( m \mid 100 \), so we could have \( m = 2, 4, 5, 10, 20, 25, 50 \) or 100.