Math 406 Section 1.1: Numbers and Sequences

1. **Numbers:**

   (a) **Definition:** Primarily in this class we'll focus on the integers:

   $$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$

   We'll also spend time on subsets of $\mathbb{Z}$ and other sets of numbers for which $\mathbb{Z}$ and $\mathbb{Z}^+$ are useful.

   (b) **Definition:** A set is *well-ordered* if every nonempty subset has a least element. Note: It's not enough that the set itself has a least element, but every subset must!

   - **Axiom:** The set $\mathbb{Z}^+$ is well-ordered.
   - **Example:** The set $\mathbb{Z}$ is not well-ordered.
   - **Example:** The set $[0, 1)$ is not well-ordered.

   (c) **Definition:** A real number is *rational* if it can be written in the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. The set of rational numbers is denoted $\mathbb{Q}$.
   Example: The real number $\sqrt{2}$ is irrational.
   Proof: There are many ways to do this. We'll do it here with well-ordering. By way of contradiction suppose $\sqrt{2}$ is rational so $\sqrt{2} = a/b$ with $a, b \in \mathbb{Z}$ and $b \neq 0$. Since $\sqrt{2} > 0$ we can also assume $a, b \in \mathbb{Z}^+$. Note $a = b\sqrt{2}$ here. Consider the set

   $$S = \left\{k \mid k \in \mathbb{Z}^+ \wedge k\sqrt{2} \in \mathbb{Z}^+\right\}$$

   Observe that $S \subset \mathbb{Z}^+$ and $b \in S$ (because $b \in \mathbb{Z}^+$ and $b\sqrt{2} = a \in \mathbb{Z}^+$) so S is nonempty and must have a least element. Let m be this least element. Consider $m' = m\sqrt{2} - m$. Consider the following facts about m':

   - $0 < m\sqrt{2} - m = m(\sqrt{2} - 1) < m$ so $0 < m' < m$.
   - Since $m \in S$ we have $m \in \mathbb{Z}^+$ and $m\sqrt{2} \in \mathbb{Z}^+$ so $m' = m\sqrt{2} - m \in \mathbb{Z}^+$.
   - Since $m \in \mathbb{Z}^+$ we have $2m \in \mathbb{Z}^+$ and since $m\sqrt{2} \in \mathbb{Z}^+$ we have $m'\sqrt{2} = (m\sqrt{2} - m)\sqrt{2} = 2m - m\sqrt{2} \in \mathbb{Z}^+$.

   It follows that $m' \in S$ but $m' < m$ which contradicts the fact that m is the least element. $\mathcal{QED}$

   (d) **Definition:** A real numbers α is *algebraic* if is is the root of a polynomial with integer coefficients. Otherwise it is *transcendental*.

   - **Example:** The real number $\sqrt{2}$ is algebraic.
   - **Example:** The real number $\sqrt{7 + \sqrt[3]{5}}$ is algebraic.
   - **Example:** The real number π is transcendental This is not at all obvious and is hard to prove.

2. **The Greatest and Least Integer Functions:**

   (a) **Definition:** For $a \in \mathbb{R}$ define the *greatest integer function* or *floor function* $\lfloor a \rfloor$ to be the greatest integer less than or equal to a.

   (b) **Definition:** For $a \in \mathbb{R}$ define the *least integer function* or *ceiling function* $\lceil a \rceil$ to be the smallest integer greater than or equal to a.

3. **Countable and Uncountable Sets:**

   (a) **Definition:** A set S is *countable* if it is either finite or if it is infinite and there is a 1-1 correspondance between S and $\mathbb{Z}^+$. Otherwise it is *uncountable*.

   - **Example:** The set $\mathbb{Z}$ is countable.
     Proof: We can arrange them as $0, 1, -1, 2, -2, 3, -3, ...$ and then line them up with $1, 2, 3, ...$
   - **Example:** The set $\mathbb{Q}$ is countable.
     Proof: We'll just do $\mathbb{Q}^+$. We arrange them in a grid as is standard...
   - **Example:** The set $[0, 1)$ is uncountable.
     Proof: Standard diagonalization argument.

4. **Sequences:** These are covered in Calculus 2 and is a prerequisite. We don't need all the convergence tests, etc., just remember what a sequence is.