Math 406 Section 1.3: Induction

1. **The (First) Principle of Mathematical Induction (Weak Induction):**

   (a) **Theorem:** Suppose a set S of positive integers contains 1 and suppose that if it contains k then it must contain $k+1$. Then $S = \mathbb{Z}^+$.
   Proof: Suppose $S \subsetneq \mathbb{Z}^+$. Consider the set $\mathbb{Z}^+ - S$ which is therefore nonempty and since it's a subset of $\mathbb{Z}^+$ it must have a least element m. We can't have $m = 1$ since $1 \in S$ thus $m > 1$ and since m is the least element in $\mathbb{Z}^+ - S$ we know $m - 1 \in S$. But since $m - 1 \in S$ we must have $m \in S$, a contradiction. $\mathcal{QED}$

   (b) **Rephrasing:** This is often phrased as follows: If a statement is true for $n = 1$ and if, when it is true for some k, it must be true for $k+1$, then it is true for all $n = 1, 2, 3, ...$

   (c) **Comment:** There's no reason to start at $n = 1$.

   (d) **Example:** If we define the Fibonnaci numbers by $f_1 = 1$, $f_2 = 1$, and $f_k = f_{k-2} + f_{k-1}$ for $k \geq 3$ then we have:
   $$\sum_{k=1}^{n} f_i = f_{n+2} - 1$$
   Proof: Blah blah... $\mathcal{QED}$

2. **The Second Principle of Mathematical Induction (Strong Induction):**

   (a) **Theorem:** Suppose a set S of positive integers contains 1 and suppose that if it contains $1, ..., k$ then it must contain $k+1$. Then $S = \mathbb{Z}^+$.
   Proof: This follows immediately from Weak Induction. $\mathcal{QED}$

   (b) **Comment 1:** The reason for having Strong Induction is that sometimes in order to prove that $k+1$ is in the set we sometimes need to rely upon the fact that something other than k is.

   (c) **Comment 2:** A fact that's not often documented or explained well (if at all) is the fact that when using Strong Induction we must often prove many base cases. For example suppose we assume $1, ..., k$ and wish to prove $k+1$. If we use $k-2$ in the process of doing this then we must be sure that $k - 2 \geq 1$ or $k \geq 3$. Thus our inductive step only works for $k \geq 3$ and we must prove base cases $n = 1, 2, 3$ at the start so we can use them as a launching point for $k = 3$. The proper way around this is usually to do the inductive step first, check the conditions, then take care of the base cases.

   (d) **Example:** Consider the claim that any postage more than 9 cents can be made out of 3-cent and 7-cent stamps. Consider the argument: Suppose we can make $9, 10, 11, ..., k$ cents. Since we can make $k-2$ cents we can add in a 3-cent stamp to make $k - 2 + 3 = k + 1$ cents. In order for this to make sense we must have $k - 2 \geq 9$ and so $k \geq 11$. Thus the base cases must include $n = 9, 10, 11$, all of which are easy to show.

   (e) **Bad Example:** Just to drive the point home let's see what happens if we ignore this base case issue. Suppose we claim that any postage more than 3 cents can be made out of 3-cent and 7-cent stamps. It's certainly true for the case $n = 3$. Suppose it's true for $3, 4, ..., k$. Since it's true for $k-2$ we add another 3-cent stamp to get $k+1$. $\mathcal{QED}$?
   Clearly not. The statement is false for $n = 4$.