Math 406 Section 3.2: Prime Numbers - How are they Distributed?

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1. **Introduction:** We may know that there are infinitely many primes but understanding how they are ditributed is a different question. As we go up through the integers are there fewer and fewer primes? Can we have any sense of what the 100 millionth prime might be?

2. **Definition:** Define $\pi(x)$ to be the number of primes less than or equal to $x \in \mathbb{R}$
   Example: $\pi(11) = 5$ because there are primes $2, 3, 5, 7, 11$.
   Example: $\pi(10) = 4$ because there are primes $2, 3, 5, 7$.
   Example: $\pi(3.2) = 2$ because there are primes $2, 3$.

3. **Theorem (The Prime Number Theorem):** We have:

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1$$

   Proof: Omitted because very hard. $\mathcal{QED}$

   Note: This states that for very large (emphasis on very) we have:

$$\pi(x) \approx \frac{x}{\ln x}$$

   Example: For example $\pi(1000000) \approx \frac{1000000}{\ln(1000000)} \approx 72382.41365054197$ In fact $\pi(1000000) = 78498$ so this is still pretty lousy.

4. **Corollary:** If p_n is the n^{th} prime then we have:

$$\lim_{x\to\infty} \frac{p_n}{n \ln n} = 1$$

Proof: From the PNT we have:

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1 \qquad \text{which may be rewritten as} \qquad \lim_{x\to\infty} \frac{\pi(x)\ln x}{x} = 1 \qquad (*)$$

If we take the natural logarithm of both sides then the rules of logarithms and the continuity of ln as well as factoring out $\ln x$ yields:

$$0 = \lim_{x\to\infty} \ln(\pi(x)) + \ln(\ln(x)) - \ln(x) = \lim_{x\to\infty} \ln(x)\left[\frac{\ln(\pi(x))}{\ln x} + \frac{\ln(\ln(x))}{\ln x} - 1\right] = 0$$

Since $\lim_{x\to\infty} \ln(\ln(x))/\ln(x) = 0$ (by Lôpital's Rule) and since $\lim_{x\to\infty} \ln(x) = \infty$, the remains of the bracketed expression must be 0 and so:

$$\lim_{x\to\infty} \frac{\ln(\pi(x))}{\ln x} = 1$$

From here we move to the discrete and observe that as $n \to \infty$ we have $p_n \to \infty$ and so we have:

$$1 = \lim_{n\to\infty} \frac{\ln(\pi(p_n))}{\ln(p_n)} = \lim_{n\to\infty} \frac{\ln(n)}{\ln(p_n)}$$

Similarly, from $(*)$ above and again noting that as $n \to \infty$ we have $p_n \to \infty$ and so we have:

$$1 = \lim_{n\to\infty} \frac{\pi(p_n)\ln(p_n)}{p_n} = \lim_{n\to\infty} \frac{n \ln(p_n)}{p_n}$$

So finally:

$$\lim_{n\to\infty} \frac{p_n}{n \ln n} = \lim_{n\to\infty} \frac{p_n}{n \ln(p_n)} \frac{\ln(p_n)}{\ln n} = (1)(1) = 1$$

QED

Note: What this states is that for very large (emphasis on very) we have:

$$p_n \approx n \ln n$$

Example: We have $p_{1M} \approx 1000000 \ln(1000000) \approx 13815510.55796427$. In fact $p_{1M} = 15485863$ so again still pretty lousy.

5. **Theorem (Gaps Between Primes):** For any n we can find n consecutive composite numbers.

   Proof: Consider the numbers:

   $$(n+1)!+2, (n+1)!+3, ..., (n+1)!+(n+1)$$

   Each of these is composite. $\mathcal{QED}$

   Note: This is not efficient at all!

6. **Conjectures:**

   (a) **Twin Prime Conjecture:** There are infinitely many pairs of primes differing by 2.

   (b) **Goldbach's Conjecture:** Every positive even integer greater than 2 can be written as the sum of two primes.

   (c) **The n^2+1 Conjecture:** There are infinitely many primes of the form n^2+1 for $n \in \mathbb{Z}$.

   (d) **The Legendre Conjecture:** There is a prime between every two pairs of consecutive squares of integers.