Math 406 Section 7.2: The Sum and Number of Divisors

1. **Introduction:** Besides Euler's ϕ function there are two other interesting arithmetic functions.

2. **Definition:** Define $\sigma(n)$ to be the sum of all positive divisors of n

   Definition: Define $\tau(n)$ to be the number of positive divisors of n.

   Notice that:

$$\sigma(n) = \sum_{d|n} d \qquad \text{and} \qquad \tau(n) = \sum_{d|n} 1$$

3. **Theorem:**

   If f is multiplicative then so is $\sum_{d|n} f(d)$. In other words if $\gcd(m,n) = 1$ then:

$$F(mn) = F(m)F(n)$$

   More specifically:

$$\sum_{d|mn} f(d) = \sum_{d|m} f(d) \sum_{d|n} f(d)$$

   Proof:

   Assume $\gcd(m,n) = 1$ and we are interested in:

$$\sum_{d|mn} f(d)$$

   Since $\gcd(m,n) = 1$ every divisor d of mn may be separated into a product $d = d_m d_n$ with $d_m \mid m$ and $d_n \mid n$ and with $\gcd(d_m, d_n) = 1$ and vice versa, if $d_m \mid m$ and $d_n \mid n$ then $d = d_m d_n$ is a divisor of mn.

   Thus:

$$\begin{aligned} \sum_{d|mn} f(d) &= \sum_{\substack{d_m|m \\ d_n|n}} f(d_m d_n) \\ &= \sum_{\substack{d_m|m \\ d_n|n}} f(d_m) f(d_n) \\ &= \sum_{d_m|m} f(d_m) \sum_{d_n|n} f(d_n) \end{aligned}$$

   $\mathcal{QED}$

   If this final step isn't clear a single example can help. If $m = 3$ (with divisors $1, 3$) and $n = 35$ (with divisors $1, 5, 7, 35$) then observe that:

$$
\begin{aligned}
\sum_{d_m|3} f(d_m) \sum_{d_n|35} f(d_n) &= [f(1)+f(3)]\,[f(1)+f(5)+f(7)+f(35)] \\
&= f(1)f(1)+f(1)f(5)+f(1)f(7)+f(1)f(35) \\
&+ f(3)f(1)+f(3)f(5)+f(3)f(7)+f(3)f(35) \\
&= \sum_{\substack{d_m|3 \\ d_n|35}} f(d_m)f(d_n)
\end{aligned}
$$

4. **Corollary:**

   σ and τ are multiplicative.

   Proof:

   Follows since $f(d) = d$ and $f(d) = 1$ are multiplicative and since $\sigma(n) = \sum_{d|n} d$ and $\tau(n) = \sum_{d|n} 1$ as we saw before.

   $\mathcal{QED}$

5. **Theorem (Calculation of σ):**

   We have $\sigma(p^\alpha) = 1 + p + ... + p^\alpha = \frac{p^{\alpha+1}-1}{p-1}$ and so:

$$\sigma(p_1^{\alpha_1}...p_k^{\alpha_k}) = \prod_{i=1}^{k} \left(1 + p + p^2 + ... + p_i^{\alpha_i}\right) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}$$

   Example:

   We have:

$$\sigma(2^3 \cdot 3 \cdot 11^2) = (1 + 2 + 2^2 + 2^3)(1 + 3)(1 + 11 + 11^2) = 7980$$

6. **Theorem (Calculation of τ):**

   We have $\tau(p^\alpha) = \alpha + 1$ and so:

$$\tau(p_1^{\alpha_1}...p_k^{\alpha_k}) = \prod_{i=1}^{k} (\alpha_i + 1)$$

   Example:

   We have:

$$\tau(2^3 \cdot 3 \cdot 11^2) = (3 + 1)(1 + 1)(2 + 1) = 24$$

7. **Note:**

   There are many ways that ϕ, σ, and τ arise. Here are a few examples:

   Example:

   Therea are no n with $\sigma(n) = 10$. This is because $\sigma(n)$ is a product of geometric sums of the form $1 + p + ... + p^\alpha$ which provides a severe restriction.

   First note that it's impossible to have $p \geq 11$ since the geometric sums would be larger than 10. Thus we could only have $p = 2, 3, 5, 7$.

   Then note that in order for the geometric sums to be less than or equal to 10:

   - If $p = 2$ the geometric sums can only be $1, 3, 7$.
   - If $p = 3$ the geometric sums can only be $1, 4$.
   - If $p = 5$ the geometric sums can only be $1, 6$.
   - If $p = 7$ the geometric sums can only be $1, 8$.

   There is no way to get a product of these equal to 10.

   Example:

   There are infinitely many n with $\tau(n) = 10$. This is because we can have, for example, $n = pq^4$ for any distinct primes p, q and $\tau(n) = (1 + 1)(4 + 1) = 10$.

   Example:

   If p is prime then $\sigma(p) = \phi(p) + \tau(p)$.

   This may seem suprising but really isn't hard to prove, since $\sigma(p) = p + 1$ and $\phi(p) = p - 1$ and $\tau(p) = 2$ and the result follows.

   There are other such relationships that arise for non-primes, there is one on the homework.