1. **Introduction**: Besides Euler's $\phi$ function there are two other interesting arithmetic functions.

2. **Definition**: Define $\sigma(n)$ to be the sum of all positive divisors of $n$
   **Definition**: Define $\tau(n)$ to be the number of positive divisors of $n$.

   Notice that:
   
   $\sigma(n) = \sum_{d \mid n} d$ \quad \text{and} \quad \tau(n) = \sum_{d \mid n} 1$

3. **Theorem**: If $f$ is multiplicative then so is $\sum_{d \mid n} f(d)$. In other words if $\gcd(m, n) = 1$ then:

   \[ \sum_{d \mid mn} f(d) = \sum_{d \mid m} f(d) \sum_{d \mid n} f(d) \]

   **Proof**: Assume $\gcd(m, n) = 1$ and we are interested in:

   \[ \sum_{d \mid mn} f(d) \]

   Since $\gcd(m, n) = 1$ every divisor $d$ of $mn$ may be separated into a product $d = d_m d_n$ with $d_m \mid m$ and $d_n \mid n$ and with $\gcd(d_m, d_n) = 1$ and vice versa, if $d_m \mid m$ and $d_n \mid n$ then $d = d_m d_n$ is a divisor of $mn$.

   Thus:

   \[
   \sum_{d \mid mn} f(d) = \sum_{d_m \mid m} f(d_m) \sum_{d_n \mid n} f(d_n) \\
   = \sum_{d_m \mid m} f(d_m) f(d_n) \\
   = \sum_{d_m \mid m} f(d_m) \sum_{d_n \mid n} f(d_n) 
   \]

   \[ \boxed{\text{QED}} \]

   If this final step isn’t clear a single example can help. If $m = 3$ (with divisors 1, 3) and $n = 35$ (with divisors 1, 5, 7, 35) then observe that:

   \[
   \sum_{d_m \mid 3} f(d_m) \sum_{d_n \mid 35} f(d_n) = [f(1) + f(3)] [f(1) + f(5) + f(7) + f(35)] \\
   = f(1)f(1) + f(1)f(5) + f(1)f(7) + f(1)f(35) + f(3)f(1) + f(3)f(5) + f(3)f(7) + f(3)f(35) \\
   = \sum_{d_m \mid 3} f(d_m) f(d_n) 
   \]
4. Corollary:
\(\sigma\) and \(\tau\) are multiplicative.

**Proof:**
Follows since \(f(d) = d\) and \(f(d) = 1\) are multiplicative and since \(\sigma(n) = \sum_{d|n} d\) and \(\tau(n) = \sum_{d|n} 1\) as we saw before.

QED

5. Theorem (Calculation of \(\sigma\)):
We have \(\sigma(p^\alpha) = 1 + p + \ldots + p^\alpha = \frac{p^{\alpha+1} - 1}{p-1}\) and so:

\[
\sigma(p_1^{\alpha_1} \ldots p_k^{\alpha_k}) = \prod_{i=1}^{k} (1 + p + p^2 + \ldots + p_i^{\alpha_i}) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}
\]

**Example:**
We have:
\[
\sigma(2^3 \cdot 3 \cdot 11^2) = (1 + 2 + 2^2 + 2^3)(1 + 3)(1 + 11 + 11^2) = 7980
\]

6. Theorem (Calculation of \(\tau\)):
We have \(\tau(p^\alpha) = \alpha + 1\) and so:

\[
\tau(p_1^{\alpha_1} \ldots p_k^{\alpha_k}) = \prod_{i=1}^{k} (\alpha_i + 1)
\]

**Example:**
We have:
\[
\tau(2^3 \cdot 3 \cdot 11^2) = (3 + 1)(1 + 1)(2 + 1) = 24
\]
Note:

There are many ways that $\phi$, $\sigma$, and $\tau$ arise. Here are a few examples:

Example:

There are no $n$ with $\sigma(n) = 10$. This is because $\sigma(n)$ is a product of geometric sums of the form $1 + p + \ldots + p^a$ which provides a severe restriction.

First note that it's impossible to have $p \geq 11$ since the geometric sums would be larger than 10. Thus we could only have $p = 2, 3, 5, 7$.

Then note that in order for the geometric sums to be less than or equal to 10:

- If $p = 2$ the geometric sums can only be 1, 3, 7.
- If $p = 3$ the geometric sums can only be 1, 4.
- If $p = 5$ the geometric sums can only be 1, 6.
- If $p = 7$ the geometric sums can only be 1, 8.

There is no way to get a product of these equal to 10.

Example:

There are infinitely many $n$ with $\tau(n) = 10$. This is because we can have, for example, $n = pq^4$ for any distinct primes $p, q$ and $\tau(n) = (1 + 1)(4 + 1) = 10$.

Example:

If $p$ is prime then $\sigma(p) = \phi(p) + \tau(p)$.

This may seem surprising but really isn’t hard to prove, since $\sigma(p) = p + 1$ and $\phi(p) = p - 1$ and $\tau(p) = 2$ and the result follows.

There are other such relationships that arise for non-primes, there is one on the homework.