1. Given $A = 1245090$ and $B = 1647750$.\[15\text{ pts}\]
   (a) Find the prime factorizations of $A$ and $B$ and use them to find $\gcd(A, B)$.
   (b) Find $\gcd(A, B)$ using the Euclidean Algorithm.

2. Use the Chinese Remainder Theorem to find the smallest and second smallest nonnegative solutions to the system: \[15\text{ pts}\]
   \[
   \begin{align*}
   x &\equiv 3 \mod 12 \\
   x &\equiv 7 \mod 13 \\
   x &\equiv 2 \mod 5
   \end{align*}
   \]

3. For each of $n = 41, 848, 4204, 85504$ find the exact value $p_n$ of the $n$th prime (however you want) and then approximate value $a_n$ of the $n$th prime (using the Prime Number Theorem Corollary). Calculate the percentage error
   \[
   \frac{100|p_n - a_n|}{p_n}
   \]
   for each. \[10\text{ pts}\]

4. Find all incongruent solutions mod 160 to the linear system: \[10\text{ pts}\]
   \[
   155x \equiv 20 \mod 160
   \]

5. Find all primitive roots for $n = 19$ as follows: First find the smallest positive primitive root. Then use the Theorem from class which yields all the remaining ones. Final answers should be least nonnegative residues. \[15\text{ pts}\]

6. It’s a fact that $r = 14$ is a primitive root mod 17. \[15\text{ pts}\]
   (a) Use this to construct a table of indices for this primitive root.
   (b) Use the table of indices to solve the equation: $x^{14} \equiv 15 \mod 17$. Your answer(s) should be mod 17.
   (c) Use the table of indices to solve the equation: $9^x \equiv 15 \mod 17$. Your answer(s) should be mod 16.

7. Calculate the following Jacobi symbols: \[15\text{ pts}\]
   (a) $\left(\frac{2423}{1277}\right)$
   (b) $\left(\frac{2423}{36593}\right)$
8. Suppose you intercept the following ciphertext from Alice to Bob:

\[ 20631 9532 9946 21611 7666 5245 18116 25960 16821 \]

You know that Bob’s public key is \((e, n) = (1663, 26167)\). Bob thinks this is secure because he doesn’t believe that his \(n\) can be factored easily. Factor \(n = 26167\), find \(\phi(n)\), find \(d\) and then decrypt the message. Be clear about the steps you take.

9. Determine if each of the following sets is well-ordered. If a set is not well-ordered give evidence. If a set is well-ordered no evidence is required.

(a) \(\{0\} \cup \left\{ \frac{n+3}{n} \left| n \in \mathbb{Z}^+ \right. \right\} \)

(b) \(5\mathbb{Z} \)

(c) \(\left\{ \lfloor \sqrt{n} \rfloor \left| n \in \mathbb{Z}^+ \right. \right\} \)

10. Suppose \(p \geq 47\) is an unknown prime. Find all solutions to \(x^2 + 45 \equiv 14x \mod p\). Note that your solutions will be mod \(p\).

11. Consider the inequality:

\[ 5^n < n! \]

(a) Find the smallest positive integer \(n_0\) for which this is true. Do this however you wish.

(b) Prove by induction that \(5^n < n!\) for all \(n \geq n_0\).

12. Suppose \(p\) is an odd prime such that there is some \(a\) so that \(a\) is a quadratic residue of \(p\) but \(2a\) is a quadratic non-residue of \(p\). Prove that \(p \equiv \pm 3 \mod 8\).

13. Prove that for \(a, b \in \mathbb{Z}\) and \(n \in \mathbb{Z}^+\) that if \(a^n \mid b^n\) then \(a \mid b\).

14. Prove that if \(a, b, c \in \mathbb{Z}\) with \(\gcd(a, b) = 1\) and \(c \mid (a + b)\) then \(\gcd(c, a) = \gcd(c, b) = 1\).