1. Given $A = 108086$ and $B = 188650$. [15 pts]
   (a) Find the prime factorizations of $A$ and $B$ and use them to find $\gcd(A, B)$.
   (b) Find $\gcd(A, B)$ using the Euclidean Algorithm.

2. Use the Chinese Remainder Theorem to find the smallest and second smallest nonnegative solutions to the system: [15 pts]
   
   \[
   x \equiv 3 \mod 4 \\
   x \equiv 1 \mod 7 \\
   x \equiv 1 \mod 11
   \]

3. For each of $n = 18, 279, 4952, 16404$ find the exact value $p_n$ of the $n^{\text{th}}$ prime (however you want) and then approximate value $a_n$ of the $n^{\text{th}}$ prime (using the Prime Number Theorem Corollary). Calculate the percentage error \[
\frac{100|p_n - a_n|}{p_n}
\]
for each. [10 pts]

4. Find all incongruent solutions mod 88 to the linear system: [10 pts]
   
   \[20x \equiv 20 \mod 88\]

5. Find all primitive roots for $n = 13$ as follows: First find the smallest positive primitive root. Then use the Theorem from class which yields all the remaining ones. Final answers should be least nonnegative residues. [15 pts]

6. It’s a fact that $r = 12$ is a primitive root mod 17. [15 pts]
   (a) Use this to construct a table of indices for this primitive root.
   (b) Use the table of indices to solve the equation: $x^6 \equiv 2 \mod 17$. Your answer(s) should be mod 17.
   (c) Use the table of indices to solve the equation: $2^x \equiv 2 \mod 17$. Your answer(s) should be mod 16.

7. Calculate the following Jacobi symbols: [15 pts]
   (a) \[
   \left( \frac{850}{667} \right)
   \]
   (b) \[
   \left( \frac{850}{28681} \right)
   \]
8. Suppose you intercept the following ciphertext from Alice to Bob: [15 pts]

\[18099 \ 8242 \ 3193 \ 4371 \ 11891 \ 2921 \ 16465 \ 9417\]

You know that Bob’s public key is \((e,n) = (1199, 18203)\). Bob thinks this is secure because he doesn’t believe that his \(n\) can be factored easily. Factor \(n = 18203\), find \(\phi(n)\), find \(d\) and then decrypt the message. Be clear about the steps you take.

9. Determine if each of the following sets is well-ordered. If a set is not well-ordered give evidence. If a set is well-ordered no evidence is required. [15 pts]

(a) \(\{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}\)

(b) \(\{0\} \cup \left\{\frac{n+3}{n} \mid n \in \mathbb{Z}^+\right\}\)

(c) \([0, \infty) \cap 2\mathbb{Z}\)

10. Suppose \(p \geq 7\) is an unknown prime. Find all solutions to \(x^2 + 6 \equiv 5x \mod p\). Note that your solutions will be mod \(p\). [15 pts]

11. Consider the inequality: [15 pts]

\[3^n < n!\]

(a) Find the smallest positive integer \(n_0\) for which this is true. Do this however you wish.

(b) Prove by induction that \(3^n < n!\) for all \(n \geq n_0\).

12. Suppose \(p\) is an odd prime such that there is some \(a\) so that \(a\) is a quadratic residue of \(p\) but \(2a\) is a quadratic non-residue of \(p\). Prove that \(p \equiv \pm 3 \mod 8\). [15 pts]

13. Prove that for \(a, b \in \mathbb{Z}\) and \(n \in \mathbb{Z}^+\) that if \(a^n \mid b^n\) then \(a \mid b\). [15 pts]

14. Prove that if \(a, b, c \in \mathbb{Z}\) with \(\gcd(a, b) = 1\) and \(c \mid (a + b)\) then \(\gcd(c, a) = \gcd(c, b) = 1\). [15 pts]