1. Given $A = 114350075$ and $B = 260718171$.  
   
   (a) Find the prime factorizations of $A$ and $B$ and use them to find $\gcd(A, B)$.  
   (b) Find $\gcd(A, B)$ using the Euclidean Algorithm.

2. Use the Chinese Remainder Theorem to find the smallest and second smallest nonnegative solutions to the system:

   \begin{align*}
   x &\equiv 9 \mod 12 \\
   x &\equiv 8 \mod 25 \\
   x &\equiv 20 \mod 29
   \end{align*}

3. For each of $n = 41, 830, 3548, 68192$ find the exact value $p_n$ of the $n^{th}$ prime (however you want) and then approximate value $a_n$ of the $n^{th}$ prime (using the Prime Number Theorem Corollary). Calculate the percentage error

   \[ \frac{100 |p_n - a_n|}{p_n} \]

   for each.

4. Find all incongruent solutions mod 115 to the linear system:

   \[ 10x \equiv 20 \mod 115 \]

5. Find all primitive roots for $n = 31$ as follows: First find the smallest positive primitive root. Then use the Theorem from class which yields all the remaining ones. Final answers should be least nonnegative residues.

6. It’s a fact that $r = 7$ is a primitive root mod 11.
   
   (a) Use this to construct a table of indices for this primitive root.
   
   (b) Use the table of indices to solve the equation: $x^6 \equiv 4 \mod 11$. Your answer(s) should be mod 11.
   
   (c) Use the table of indices to solve the equation: $5^x \equiv 4 \mod 11$. Your answer(s) should be mod 10.

7. Calculate the following Jacobi symbols:
   
   (a) $\left( \frac{2170}{1189} \right)$
   
   (b) $\left( \frac{2170}{30659} \right)$
8. Suppose you intercept the following ciphertext from Alice to Bob: [15 pts]

\[16823 \ 14055 \ 7802 \ 14850 \ 17379 \ 6373\]

You know that Bob’s public key is \((e, n) = (1441, 20453)\). Bob thinks this is secure because he doesn’t believe that his \(n\) can be factored easily. Factor \(n = 20453\), find \(\phi(n)\), find \(d\) and then decrypt the message. Be clear about the steps you take.

9. Determine if each of the following sets is well-ordered. If a set is not well-ordered give evidence. [15 pts]
If a set is well-ordered no evidence is required.

(a) \(\{0\} \cup \left\{ \frac{n+3}{n} \mid n \in \mathbb{Z}^+ \right\}\)

(b) \(\{0, \frac{1}{3}, \frac{1}{2}, \frac{1}{3^2}, \ldots \}\)

(c) \(\left\{ \lfloor \sqrt{n} \rfloor \mid n \in \mathbb{Z}^+ \right\}\)

10. Suppose \(p \geq 47\) is an unknown prime. Find all solutions to \(x^2 + 45 \equiv 14x \mod p\). Note that your solutions will be mod \(p\). [15 pts]

11. Consider the inequality: [15 pts]

\[5^n < n!\]

(a) Find the smallest positive integer \(n_0\) for which this is true. Do this however you wish.

(b) Prove by induction that \(5^n < n!\) for all \(n \geq n_0\).

12. Suppose \(p\) is an odd prime such that there is some \(a\) so that \(a\) is a quadratic residue of \(p\) but \(2a\) is a quadratic non-residue of \(p\). Prove that \(p \equiv \pm 3 \mod 8\). [15 pts]

13. Prove that for \(a, b \in \mathbb{Z}\) and \(n \in \mathbb{Z}^+\) that if \(a^n \mid b^n\) then \(a \mid b\). [15 pts]

14. Prove that if \(a, b, c \in \mathbb{Z}\) with \(\gcd(a, b) = 1\) and \(c \mid (a + b)\) then \(\gcd(c, a) = \gcd(c, b) = 1\). [15 pts]