1. Given $A = 18266534$ and $B = 414464050$. [15 pts]
   (a) Find the prime factorizations of $A$ and $B$ and use them to find $\gcd(A, B)$.
   (b) Find $\gcd(A, B)$ using the Euclidean Algorithm.

2. Use the Chinese Remainder Theorem to find the smallest and second smallest nonnegative solutions to the system: [15 pts]
   
   $x \equiv 23 \mod 29$
   $x \equiv 7 \mod 27$
   $x \equiv 6 \mod 11$

3. For each of $n = 96, 876, 5222, 24253$ find the exact value $p_n$ of the $n^{th}$ prime (however you want) and then approximate value $a_n$ of the $n^{th}$ prime (using the Prime Number Theorem Corollary). Calculate the percentage error
   
   $\frac{100|p_n - a_n|}{p_n}$
   
   for each. [10 pts]

4. Find all incongruent solutions mod 128 to the linear system: [10 pts]
   
   $72x \equiv 96 \mod 128$

5. Find all primitive roots for $n = 31$ as follows: First find the smallest positive primitive root. Then use the Theorem from class which yields all the remaining ones. Final answers should be least nonnegative residues. [15 pts]

6. It’s a fact that $r = 11$ is a primitive root mod 13. [15 pts]
   (a) Use this to construct a table of indices for this primitive root.
   (b) Use the table of indices to solve the equation: $x^2 \equiv 12 \mod 13$. Your answer(s) should be mod 13.
   (c) Use the table of indices to solve the equation: $10^x \equiv 12 \mod 13$. Your answer(s) should be mod 12.

7. Calculate the following Jacobi symbols: [15 pts]
   (a) $\left(\frac{2334}{1337}\right)$
   (b) $\left(\frac{2334}{26381}\right)$
8. Suppose you intercept the following ciphertext from Alice to Bob: [15 pts]

\[
\begin{array}{c}
1708 \\ 5200 \\ 23286 \\ 29963 \\ 17094 \\ 4922 \\ 20152
\end{array}
\]

You know that Bob’s public key is \((e, n) = (1863, 37627)\). Bob thinks this is secure because he doesn’t believe that his \(n\) can be factored easily. Factor \(n = 37627\), find \(\phi(n)\), find \(d\) and then decrypt the message. Be clear about the steps you take.

9. Determine if each of the following sets is well-ordered. If a set is not well-ordered give evidence. [15 pts]

If a set is well-ordered no evidence is required.

(a) \(\{0\} \cup \left\{ \frac{n+3}{n} \mid n \in \mathbb{Z}^+ \right\} \)

(b) \(\{0, \frac{1}{3}, \frac{1}{25}, \frac{1}{125}, ...\} \)

(c) \(\{0\} \cup \left\{ \frac{n}{n+3} \mid n \in \mathbb{Z}^+ \right\} \)

10. Suppose \(p \geq 97\) is an unknown prime. Find all solutions to \(x^2 + 90 \equiv 19x \mod p\). Note that your solutions will be mod \(p\). [15 pts]

11. Consider the inequality: [15 pts]

\[9^n \lt n!\]

(a) Find the smallest positive integer \(n_0\) for which this is true. Do this however you wish.

(b) Prove by induction that \(9^n \lt n!\) for all \(n \geq n_0\).

12. Suppose \(p\) is an odd prime such that there is some \(a\) so that \(a\) is a quadratic residue of \(p\) but \(2a\) is a quadratic non-residue of \(p\). Prove that \(p \equiv \pm 3 \mod 8\). [15 pts]

13. Prove that for \(a, b \in \mathbb{Z}\) and \(n \in \mathbb{Z}^+\) that if \(a^n \mid b^n\) then \(a \mid b\). [15 pts]

14. Prove that if \(a, b, c \in \mathbb{Z}\) with \(\gcd(a, b) = 1\) and \(c \mid (a + b)\) then \(\gcd(c, a) = \gcd(c, b) = 1\). [15 pts]