Math 410 Section 2.1: Sequences and Convergence

1. **Definition:** A sequence is formally a function $f : \mathbb{N} \to \mathbb{R}$.
   
   **Example:** $f(n) = n^2$.
   
   We think of a sequence as a succession of terms, though, like if we plugged in $1, 2, 3, \ldots$ so the above would be $1, 4, 9, 16, \ldots$.
   
   More typical notation would be one of $a_n = n^2$ or $\{n^2\}$ or giving the terms if it’s clear.
   
   Sometimes a sequence may be given recursively.

   **Example:** If $a_1 = 4$ and for $n > 1$ we have $a_n = \sqrt{a_{n-1} + 2}$. Then $a_2 = \sqrt{a_1 + 2} = \sqrt{6}$ and $a_3 = \sqrt{a_2 + 2} = \sqrt{6 + 2}$ and so on.

2. **Convergence**
   
   (a) **Idea:** We are interested in the long-term behavior of a sequence as $n \to \infty$. For example the terms in the sequence $\{\frac{1}{n}\}$ approach $0$ while the terms in the sequence $\{2^n\}$ head off to infinity.
   
   The specific case when the terms in a sequence $\{a_n\}$ approach a specific value $a$ is captured by the following idea that we then formalize:

   "No matter how close we want the terms to get to $a$, eventually they get that close and stay that close."

   (b) **Definition:** We define $\{a_n\} \to a$ if:

   \[
   \forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that if } n \geq N \text{ then } |a_n - a| < \epsilon
   \]

   In practice when we’re using this to show convergence we have to obey the quantifiers - we start with an (unknown) $\epsilon$ and show how we can get some $N$ (which will almost always depend on $\epsilon$) so that for $n \geq N$ we have $|a_n - a| < \epsilon$.

   **Example:**
   
   Show $\left\{\frac{3}{n}\right\} \to 0$.
   
   Scratch: Assume $\epsilon > 0$ is given (and unknown). We need $N$ so that if $n \geq N$ then $\left|\frac{3}{n} - 0\right| < \epsilon$.
   
   Notice that $\left|\frac{3}{n} - 0\right| = \frac{3}{n}$ and $\frac{3}{n} < \epsilon$ iff $n > \frac{3}{\epsilon}$ so as long as $N > \frac{3}{\epsilon}$ we’re good.
   
   Proof: Given $\epsilon > 0$ let $N = \left\lceil \frac{3}{\epsilon} \right\rceil + 1$. Then if $n \geq N$ then $n \geq \left\lceil \frac{3}{\epsilon} \right\rceil > \frac{3}{\epsilon}$ and so $\left|\frac{3}{n} - 0\right| < \epsilon$.

   **Example:**
   
   Show $\left\{\frac{7}{n^2} - \frac{2}{n} + 5\right\} \to 5$.
   
   Scratch: Observe that using the Triangle Inequality

   \[
   \left|\frac{7}{n^2} - \frac{2}{n} + 5 - 5\right| = \left|\frac{7}{n^2} + \left(-\frac{2}{n}\right)\right| \leq \left|\frac{7}{n^2}\right| + \left|\frac{2}{n}\right| \leq \left|\frac{7}{n}\right| + \left|\frac{2}{n}\right| = \frac{9}{n}
   \]

   Since $\frac{9}{n} < \epsilon$ iff $n > \frac{9}{\epsilon}$ we know as long as $N > \frac{9}{\epsilon}$ we’re good.
   
   Proof: Given $\epsilon > 0$ let $N = \left\lceil \frac{9}{\epsilon} \right\rceil + 1$. Then if $n \geq N$ then $n \geq \left\lceil \frac{9}{\epsilon} \right\rceil + 1 > \frac{9}{\epsilon}$ and so

   \[
   \left|\frac{7}{n^2} - \frac{2}{n} + 5 - 5\right| = \left|\frac{7}{n^2} + \left(-\frac{2}{n}\right)\right| \leq \left|\frac{7}{n^2}\right| + \left|\frac{2}{n}\right| \leq \left|\frac{7}{n}\right| + \left|\frac{2}{n}\right| = \frac{9}{n} < \epsilon
   \]
Example:
Prove that if \( \{a_n\} \to 2 \) then \( \left\{ \frac{1}{a_n} \right\} \to \frac{1}{2} \).

The claim here is that:
\[
\forall \epsilon > 0 \ \exists N \in \mathbb{N} \text{ st if } n \geq N \text{ then } \left| \frac{1}{a_n} - \frac{1}{2} \right| < \epsilon
\]

Scratch
So given some unknown \( \epsilon > 0 \) how can we choose \( N \in \mathbb{N} \) so that if \( n \geq N \) then \( \left| \frac{1}{a_n} - \frac{1}{2} \right| < \epsilon \)?

Observe that
\[
\left| \frac{1}{a_n} - \frac{1}{2} \right| = \left| \frac{2 - a_n}{2a_n} \right| = \left| \frac{a_n - 2}{2a_n} \right|
\]

so really we’re trying to make \( \left| \frac{a_n - 2}{2a_n} \right| \) small ... less than \( \epsilon \).

We know that we can make \( a_n \) as close to 2 as we like because \( \{a_n\} \to 2 \) so we can make the numerator as small as we like but that denominator is awkward. We know it’s approaching 4 because \( \{a_n\} \to 2 \) but we don’t have an inequality for it - it could be bigger or smaller than 4 for any given \( n \).

However since \( \{a_n\} \to 2 \) we know that eventually \( a_n > 1 \) because eventually \( \{a_n\} \) is as close as we like to 2. If \( a_n > 1 \) then we’d have:
\[
\left| \frac{a_n - 2}{2a_n} \right| < \frac{1}{2} |a_n - 2|
\]

at this point we can make \( |a_n - 2| < 2\epsilon \) and we have what we want.

Note that we need two cutoffs here. We need \( N_1 \) beyond which \( a_n > 1 \) and \( N_2 \) beyond which \( |a_n - 2| < 2\epsilon \).

Formal Proof
Given \( \epsilon > 0 \):
- Choose \( N_1 \) so that if \( n \geq N_1 \) then \( |a_n - 2| < 1 \) so that \( a_n > 1 \).
- Choose \( N_2 \) so that if \( n \geq N_2 \) then \( |a_n - 2| < 2\epsilon \).

Let \( N = \max\{N_1, N_2\} \). Then if \( n \geq N \) then we have:
\[
\left| \frac{1}{a_n} - \frac{1}{2} \right| = \frac{2 - a_n}{2a_n} = \left| \frac{a_n - 2}{2a_n} \right| < \frac{1}{2} |a_n - 2| < \frac{1}{2}(2\epsilon) = \epsilon
\]

QED
3. The Comparison Lemma

The Comparison Lemma is a very useful tool for showing convergence of one sequence based on convergence of another.

**Theorem (The Comparison Lemma):**

Suppose \( \{a_n\} \to a \) and suppose \( \{b_n\} \) is a sequence and \( b \in \mathbb{R} \). Now suppose there is some \( C \in \mathbb{R}^+ \) and some \( N \in \mathbb{N} \) such that if \( n \geq N \) then \( |b_n - b| < C|a_n - a| \). Then \( \{b_n\} \to b \).

**Intuition:** We need to get \( |b_n - b| \) small. We go far enough in the sequence for the inequality to be true and far enough for \( C|a_n - a| \) to be small enough.

**Proof:**

Let \( \epsilon > 0 \). Choose \( N_1 \) so that if \( n \geq N_1 \) then \( |b_n - b| < \frac{\epsilon}{C} \). Let \( N = \max\{N_1, N_2\} \). Then if \( n \geq N \) then

\[
|b_n - b| < C|a_n - a| < C \left( \frac{\epsilon}{C} \right) = \epsilon
\]

QED

4. Theorem (Combinations):

If \( \{a_n\} \to a \) and \( \{b_n\} \to b \) then:

(a) \( \{a_n \pm b_n\} \to a \pm b \)

**Proof for +:** Let \( \epsilon > 0 \). Choose \( N_1 \) so that if \( n \geq N_1 \) then \( |a_n - a| < \frac{\epsilon}{2} \) and choose \( N_2 \) so that if \( n \geq N_2 \) then \( |b_n - b| < \frac{\epsilon}{2} \). Let \( N = \max\{N_1, N_2\} \) then if \( n \geq N \) then

\[
|(a_n + b_n) - (a + b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| < \epsilon
\]

QED

(b) \( \{a_n b_n\} \to ab \)

(c) \( \left\{ \frac{a_n}{b_n} \right\} \to \frac{a}{b} \) provided \( b \neq 0 \) and \( \forall n, b_n \neq 0 \).

(d) If \( p(x) \) is a polynomial then \( \{p(a_n)\} \to p(a) \).