1. **Intermediate Value Theorem:** Suppose \( f : [a, b] \to \mathbb{R} \) is continuous and \( c \) is strictly between \( f(a) \) and \( f(b) \) then there exists some \( x_0 \in (a, b) \) such that \( f(x_0) = c \).

**Proof:** Note that if \( f(a) = f(b) \) then there is no such \( c \) so we only need to consider \( f(a) < c < f(b) \) and \( f(a) > c > f(b) \). Look at the case \( f(a) < c < f(b) \).

We’re going to use the Bisection Method to construct two sequences as follows:

Define \( a_1 = a \) and \( b_1 = b \). Then look at \( \frac{a_1 + b_1}{2} \) (the midpoint) and check:

- If \( f \left( \frac{a_1 + b_1}{2} \right) \leq c \) define \( a_2 = \frac{a_1 + b_1}{2} \) and \( b_2 = b_1 \).
- If \( f \left( \frac{a_1 + b_1}{2} \right) > c \) define \( a_2 = a_1 \) and \( b_2 = \frac{a_1 + b_1}{2} \).

We then repeat the procedure looking at the midpoint of \([a_2, b_2]\) and defining \( a_3 \) and \( b_3 \) accordingly and so on, to define sequences \( \{a_n\} \) and \( \{b_n\} \).

Observe that \( \{a_n\} \) is monotone increasing and bounded above by \( b \) and \( \{b_n\} \) is monotone decreasing and bounded below by \( a \). It follows that both converge. Moreover since

\[
\frac{b_n - a_n}{2} = \frac{1}{2^n}(b - a)
\]

we know that the difference converges to 0 and so they both converge to the same value, call it \( x_0 \). That is, \( \{a_n\} \to x_0 \) and \( \{b_n\} \to x_0 \). We then claim that \( f(x_0) = c \).

From continuity we have \( \{f(a_n)\} \to f(x_0) \) and \( \{f(b_n)\} \to f(x_0) \) and since for all \( n \) we have \( f(a_n) \leq c \) we must have \( f(x_0) \leq c \) and since for all \( n \) we have \( f(b_n) > c \) we must have \( f(x_0) \geq c \).

Thus \( f(x_0) = c \).

The proof for \( f(a) > c > f(b) \) is similar.

\[QED\]

2. **Examples:**

Example: Consider \( f : [1, 5] \to \mathbb{R} \) given by \( f(x) = x^2 + 4x - \frac{1}{2} \). Observe that \( f(1) = 4 \) and \( f(5) = 44.8 \).

Since \( c = 10 \) is strictly between 4 and 44.8 we know there is some \( x_0 \in (1, 5) \) with \( f(x_0) = 10 \).