1. Derivatives of Inverses

(a) **Theorem:** Let $I$ be a neighborhood of $x_0$ and let $f : I \to \mathbb{R}$ be strictly monotone and continuous. Suppose $f$ is differentiable at $x_0$ and that $f'(x_0) \neq 0$. Define $y_0 = f(x_0)$ then $f^{-1} : f(I) \to \mathbb{R}$ is differentiable at $y_0$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

**Proof:** By the IVT we know $f(I)$ is an interval hence a neighborhood of $y_0$. Let $\{y_n\}$ be a sequence in $f(I) - \{y_0\}$ converging to $y_0$. Thus for each $n$ we have $y_n = f(x_n)$ for some $x_n \in I - \{x_0\}$ (missing $x_0$ because $f$ is strictly monotone).

Observe that $\{y_n\} \to y_0$ and so by continuity of $f^{-1}$ (previous theorem - a strictly monotone function on an interval has a continuous inverse) we know that $\{f^{-1}(y_n)\} \to f^{-1}(y_0)$ which is $\{x_n\} \to x_0$ and so

$$\left\{ \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} \right\} = \left\{ \frac{x_n - x_0}{f(x_n) - f(x_0)} \right\} \to \frac{1}{f'(x_0)}$$

(b) **Theorem:** For $n \in \mathbb{N}$ define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = x^{1/n}$. Then $f$ is continuous and

$$f'(x) = \frac{1}{n} x^{1/n - 1}$$

**Proof:** Define $g(x) = x^n$ and apply the above theorem to $g$ noting that $f = g^{-1}$.
2. Derivatives of Compositions

(a) **Theorem (The Chain Rule):** Let $I$ be a neighborhood of $x_0$ and suppose $f : I \rightarrow \mathbb{R}$ is differentiable at $x_0$. Let $J$ be an open neighborhood such that $f(I) \subseteq J$ and suppose that $g : J \rightarrow \mathbb{R}$ is differentiable at $f(x_0)$. Then $g \circ f : I \rightarrow \mathbb{R}$ is differentiable at $x_0$ and

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$$

**Proof:** Let $\{x_n\}$ be a sequence in $I - \{x_0\}$ with $x_n \rightarrow x_0$. Let’s examine:

$$\left\{ \frac{g(f(x_n)) - g(f(x_0))}{x_n - x_0} \right\}$$

Observe that we have the following equality $A = BC$ where expression $B$ is defined in cases according to $n$:

$$\frac{g(f(x_n)) - g(f(x_0))}{x_n - x_0} = \begin{cases} \frac{g(f(x_n)) - g(f(x_0))}{f(x_n) - f(x_0)} & \text{if } f(x_n) - f(x_0) \neq 0 \\ g'(f(x_0)) & \text{if } f(x_n) - f(x_0) = 0 \end{cases} \begin{cases} f(x_n) - f(x_0) \\ x_n - x_0 \end{cases}$$

The reason these are equal is:

- If $f(x_n) - f(x_0) \neq 0$ then the denominator of $B$ and numerator of $C$ cancel, yielding $A = BC$.
- If $f(x_n) - f(x_0) = 0$ then $B$ equals $g'(f(x_0))$ and the numerator of $C = 0$ so $BC = 0$ but also $g(f(x_n)) - g(f(x_0)) = 0$ so $A = 0$ and so $A = BC$.

Now check out the limits of each of $B$ and $C$:

- For $C$ we have $\lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = f'(x_0)$ since $f$ is differentiable at $x_0$.
- For $B$ notice that terms with $f(x_n) - f(x_0) = 0$ equal $g'(f(x_0))$ and terms with $f(x_n) - f(x_0) \neq 0$ approach $g'(f(x_0))$ because $\{x_n\} \rightarrow x_0$ along with $f$ being continuous (since it’s differentiable), implies $\{f(x_n)\} \rightarrow f(x_0)$. All together we see that the limit is $g'(f(x_0))$.

Thus all together the limit of the right side is $g'(f(x_0))f'(x_0)$ and so:

$$\left\{ \frac{g(f(x_n)) - g(f(x_0))}{x_n - x_0} \right\} \rightarrow g'(f(x_0))f'(x_0)$$

which is what we wished to prove.