1. The Mean Value Theorem

(a) **Introduction:** The MVT is far more important than you may suspect. It provided a fundamental connection between a function and its derivative and it underlies many properties of the integral. However first we need some preliminaries.

(b) **Lemma:** Let $I$ be a neighborhood of $x_0$ and suppose $f : I \to \mathbb{R}$ is differentiable at $x_0$. If $x_0$ is either a maximizer or minimizer for $f$ then $f'(x_0) = 0$

**Proof** Suppose $f$ is a maximizer. Since $f$ is differentiable at $x_0$ and since $I$ is a neighborhood choose a sequence $\{x_0 + 1/n\}$ for $n$ starting large enough to be in $I$. we know that:

$$\left\{ \frac{f(x_0 + 1/n) - f(x_0)}{x_0 + 1/n - x_0} \right\} \to f'(x_0)$$

Since $x_0$ is a maximizer the numerator is nonpositive and so the terms of the sequence are in $(-\infty, 0]$ which is closed we must have $f'(x_0) \leq 0$. A similar argument with $\{x_0 - 1/n\}$ shows that $f'(x_0) \geq 0$ and so $f'(x_0) = 0$. The proof for a minimizer is similar.

(c) **Rolle’s Theorem:** Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Moreover assume $f(a) = f(b)$. Then there is a point $x_0 \in (a, b)$ at which $f'(x_0) = 0$.

**Proof:** By the EVT we know $f$ assumes both a maximum and a minimum. If both occur at endpoints then the function is constant and so $f'(x) = 0$ everywhere. Otherwise let $x_0$ be either a maximizer or minimizer not at an endpoint then since $(a, b)$ is a neighborhood of $x_0$ the lemma tells us that $f'(x_0) = 0$.

(d) **The Mean Value Theorem:** Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Then there is a point $x_0 \in (a, b)$ at which

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

**Proof:** Define $h(x) = f(x) - \left[ \frac{f(b) - f(a)}{b - a} \right] x$. Observe that

$$h(a) = f(a) - \left[ \frac{f(b) - f(a)}{b - a} \right] a = f(a)(b - a) - af(b) + af(a) = \frac{bf(a) - af(b)}{b - a}$$

and

$$h(b) = f(b) - \left[ \frac{f(b) - f(a)}{b - a} \right] b = f(b)(b - a) - bf(b) + bf(a) = \frac{bf(a) - af(b)}{b - a}$$

So we can apply Rolle’s Theorem to obtain a point $x_0 \in (a, b)$ with $h'(x_0) = 0$. However then

$$0 = h'(x_0) = f'(x_0) - \left[ \frac{f(b) - f(a)}{b - a} \right]$$

and we’re done.
2. Consequences

(a) The Identity Criterion Lemma: Let $I$ be an open interval and suppose $f : I \to \mathbb{R}$ is differentiable. Then $f$ is constant iff $f'(x) = 0$ for all $x \in I$.

**Proof:** If $f$ is constant then the definition of the derivative shows that $f'(x) = 0$ for all $x \in I$.

Suppose that $f'(x) = 0$ for all $x \in I$. If $f$ is not constant then there are $a < b$ in $I$ with $f(a) \neq f(b)$. Since $f$ is differentiable on $I$ it is continuous on $[a, b]$ and differentiable on $(a, b)$ and hence by the MVT there is some $x_0 \in (a, b)$ with

$$f(x_0) = \frac{f(b) - f(a)}{b - a} \neq 0$$

which is a contradiction. Therefore $f$ is constant.

(b) The Identity Criterion: Let $I$ be an open interval and suppose $f, g : I \to \mathbb{R}$ are differentiable. Then $f$ and $g$ differ by a constant iff $f'(x) = g'(x)$ for all $x \in I$.

**Proof:** Define $g(x) = f(x) - g(x)$ and apply the ICL.

(c) Strict Monotonicity Criterion: Let $I$ be an open interval and suppose $f : I \to \mathbb{R}$ is differentiable. Suppose $f'(x) > 0$ for all $x \in I$. Then $f$ is strictly increasing. Likewise suppose $f'(x) < 0$ for all $x \in I$. Then $f$ is strictly decreasing.

**Proof:** For the $f'(x) > 0$ case suppose $a < b$ are in $I$. Since $f$ is differentiable on $I$ it is continuous on $[a, b]$ and differentiable on $(a, b)$ and hence by the MVT there is some $x_0 \in (a, b)$ with

$$\frac{f(b) - f(a)}{b - a} = f'(x_0) > 0$$

and the result follows. The $f'(x) < 0$ case is similar.